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THE DETERMINATION OF LATENT ROOTS AND INVARIANT MANIFOLDS OF MATRICES BY MEANS OF ITERATIONS[1]

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In the practical solution of many problems of vibration theory the greatest difficulty is occasioned by the necessity of solving the so-called characteristic equation. Among many methods long since suggested for the solution of this and other associated problems, one of the most effective and widely used is the method of iteration.

Given that this method is to be utilized, the presence in the matrix in hand of characteristic roots moduli of which are equal, as well as the character of the invariant manifolds of the matrix, substantially influence the course of the process of successive approximations. In works devoted to the method of iteration, a full analysis has not, up to the present, been adduced of all cases possible under the circumstances.

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[1] Prikladnai͡a matematika i mekhanika (Institute of Mechanics, USSR Academy of Sciences, Applied Mathematics and Mechanics), vol. 7 (1943), pp. 293-322 (original in Russian, with English summary).

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R. Mises and H. Pollaczek-Geiringer[1] limit themselves to symmetric matrices having no multiple latent roots. Duncan and Collar[2,3] admit the presence of a single pair of conjugate complex latent roots of maximum modulus[1] or of two latent roots of coincident moduli. The latter case is considered by passing to the limit, in connection with which the possibility of a coincidence of latent roots without an elevation of the degree of the corresponding elementary divisors is not taken into consideration.

The method of iteration is much more thoroughly investigated in a work of Aitken[4], which contains in part results identical with those of § 5 and 7 of the present work[2].

In the first part of the present work (§ 1-9) a complete analysis is carried through of all possible cases of the course of iterations, and instructions are given for practical methods of determining the dominant latent roots, as well as the corresponding invariant manifolds of the matrix, for each distinct case. Essentially the investigation is carried out by the geometrical method. A familiarity with the fundamentals of the theory of elementary divisors and of n-dimensional Euclidean geometry is required of the reader. (See, e.g., [5,6,7].)

The second part of the work gives, for the most general case, a method for a successive determination of the latent roots, in

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[1] (Translator's note) "The latent root/s of greatest modulus" (Russian) is/are spoken of as "the dominant latent root/s" (English).

[2] Aitken's work became known to the author only after completion of the present study.

order of diminishing moduli, and of the corresponding invariant manifolds. The procedure is a generalization of the method proposed for the simplest case by Duncan and Collar[2,3] and also, independently of them, by P. F. Papkovich[1].

§ 1. BASIC CONCEPTS

Let there be given a matrix

$$D = \left\{ d_{ik} \right\}, \qquad i,k = 1,2, \cdots, n$$

with real elements. In n-dimensional Euclidean space with a given orthogonal basis this matrix defines a linear transformation that relates to any vector $\underset{\sim}{a}(a_1, \cdots, a_n)$ the vector $\underset{\sim}{b} = D\underset{\sim}{a}$, the coordinates of which, $b_1, \cdots, b_n$ are defined by the formulae

$$b_i = \sum_{i=1}^{n} d_{ik} a_k \tag{1.1}$$

The vector $\underset{\sim}{f}_r(\alpha_1^{\,r}, \alpha_2^{\,r}, \cdots, \alpha_n^{\,r})$ is called the latent vector corresponding to the latent root λ_r, if

$$D\underset{\sim}{f}_r = \lambda_r \underset{\sim}{f}_r \tag{1.2}$$

Since the vector $\underset{\sim}{f}_r$ is by this equation defined only up to a constant multiplier, we shall, unless it is otherwise specified, henceforth consider the vector $\underset{\sim}{f}_r$ normalized so that

$$\underset{\sim}{f}_r|^2 = \sum_k (\alpha_k^r)^2 = 1$$

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[1] It should be noted that the conclusions of Papkovich's work are suitable only to a symmetric matrix whose latent roots are without coincident moduli, rather than to an arbitrary matrix with real latent roots as the author states.

The latent roots of a matrix are the roots of the so-called characteristic equation

$$\begin{vmatrix} d_{11} - \lambda & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} - \lambda & \cdots & d_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} - \lambda \end{vmatrix} = |D - \lambda E| = 0 \qquad (1.3)$$

where E designates the diagonal unit matrix. To simplify the exposition we shall consider the matrix D to be non-singular, i.e., not having null latent roots ($|D| \neq 0$). Equation (1.3) has n roots, $\lambda_1, \lambda_2, \cdots, \lambda_n$, which we may enumerate in order of descending moduli, i.e.,

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

Among these there may of course be equal roots.

If a latent root λ_r be known, then the determination of the corresponding latent vector reduces to the solution of the system of linear homogeneous equations

$$\sum_{k=1}^{n} (d_{ik} - \lambda_r \delta_{ik}) \alpha_k^r = 0 \qquad (i = 1, 2, \cdots, n) \quad (1.4)$$

δ_{ik} being Kronecker's delta.

We introduce into the subsequent discussion, in addition to the latent vectors, the invariant manifolds of a matrix. A linear[1]

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[1] Since only linear manifolds are considered in this work, the word linear will in future be dropped.

manifold of k ($k < n$) dimensions, E_k lying in E_n is called invariant if a vector transformed by means of the matrix D from any vector belonging to E_k also belongs to E_k.

The **simplest** example of an invariant manifold is a straight line whose direction coincides with that of a latent vector.

The structure of the invariant manifolds of a matrix in real space E_n essentially depends on the elementary divisors of the matrix D, as also upon the presence of complex roots.

§2. THE INVARIANT MANIFOLDS IN THE CASE OF REAL LATENT ROOTS

Let us first suppose that all the latent roots are real. Here, as is known, the following cases are possible.

1. All the latent roots are distinct; to each latent root corresponds a determinate latent vector. Any linear manifold E_k constructed on k latent vectors is invariant; conversely, any k-dimensional invariant manifold contains k independent latent vectors which may be taken as a basis for this manifold.

2. There are multiple latent roots, but the elementary divisors[1] of the matrix $D - \lambda E$ are relatively prime. To each latent root λ_i of multiplicity r_i there corresponds an r_i-dimensional invariant manifold E_{r_i} containing a unique invariant direction. For a basis of this E_{r_i} there may be taken the system of vectors $\underset{\sim}{f}_1^{\,i}, \underset{\sim}{f}_2^{\,i}, \cdots, \underset{\sim}{f}_{r_i}^{\,i}$ having the following properties:

$$D\underset{\sim}{f}_1^{\,i} = \lambda_i \underset{\sim}{f}_1^{\,i}, \quad D\underset{\sim}{f}_\chi^{\,i} = \lambda_i \underset{\sim}{f}_\chi + \underset{\sim}{f}_{\chi-1}, \quad \chi = 2, 3, \cdots, r_i \qquad (2.1)$$

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[1] In this work the treatment of elementary divisors follows Weierstrass.

We may call these vectors the fundamental basis (основной базис) of the invariant manifold E_{r_i} that corresponds to the latent root λ_i.

For simple latent roots the fundamental basis for each latent root will consist of a unique vector.

All the fundamental bases corresponding to the distinct latent roots contain in the aggregate n vectors, which are linearly independent and may serve as a basis of E_k.

From equality (2.1) it follows that a linear manifold E_γ constructed from the vectors $\underset{\sim}{f}_1{}^i, \underset{\sim}{f}_2{}^i, \cdots, \underset{\sim}{f}_\gamma{}^i$ ($\gamma \leq r_i$) will be an invariant manifold.

In this way we may have here a system of invariant manifolds lying one within the other, of two, three and so forth on up to r_i dimensions, containing a unique invariant direction $\underset{\sim}{f}_1{}^i$. Invariant manifolds of this type we shall call axial (осевое) and $\underset{\sim}{f}_1{}^i$ the axis (ось) of this manifold.

Apart from axial invariant manifolds, in this case it is possible to construct invariant manifolds taking as a basis an arbitrary number of each of the first vectors of the fundamental basis corresponding to the distinct latent roots. With this all invariant manifolds of the matrix D have been exhausted.

3. The elementary divisors of the matrix $D - \lambda E$ have common divisors.

For example, let there be for a certain latent root λ_h the corresponding elementary divisors

$$(\lambda - \lambda_h)^{e_1}, (\lambda - \lambda_h)^{e_2}, \cdots, (\lambda - \lambda_h)^{e_k}, \quad e_1 \leq e_2 \leq \cdots \leq e_k$$

We shall call

$$e_k = p, \qquad \sum_{i=1}^{k} e_i = s_h$$

The number of elementary divisors with exponent j we shall designate by m_j $(1 \leq j \leq p)$ and

$$\sum_{j=\chi}^{j=p} m_j = k_\chi, \quad \chi = 1, 2, \cdots, p, \quad k_1 = k$$

Note that of course not all m_j are different from zero.

Then to the latent root λ_h there corresponds a s_h-dimensional invariant manifold E_{s_h} with structure as follows: there is a k_1-dimensional manifold E_{k_1} in which each direction is invariant. In E_{k_1} lies a k_2-dimensional manifold E_{k_2} such that through each direction in it passes a two-dimensional invariant plane not belonging to E_{k_1}.

Moreover, to each number of domensions $\chi \leq p$ there corresponds a manifold of invariant directions $E_{k_\chi} \subset E_{k_1}$ such that each direction in it is an axis of determinate axial invariant manifolds placed one within the other, these manifolds being of 2, 3 and so forth up to χ dimensions and intersecting E_{k_1} only along the axis. In the light of this, obviously $E_{k_\chi} \subset E_{k_j}$ if $\chi > j$. All these invariant manifolds, ending with the p-dimensional, completely fill up a linear manifold of s_h dimensions, since $\Sigma_{j=1}^{p} jm_j = s_h$. Any vector in this E_{s_h} belongs to a certain axial invariant manifold of dimension not greater than p. As a basis of this manifold it is always possible to select a system of independent vectors

satisfying equations analogous to equations (2.1). Such a system we shall also call a fundamental basis of the given axial manifold.

In the simplest case, when $\chi_1 = \chi_2 = \cdots = \chi_k = 1$, we shall have a k-dimensional manifold E_k in which each direction will be invariant. Here $E_{k_1} \equiv E_{s_h}$.

In the general case to each latent root λ_h will correspond an invariant manifold $E_{s_h} (\sum_h s_h = n)$, of the structure indicated above.

An arbitrary invariant manifold of the matrix D may be constructed upon the independent axial invariant manifolds[1], the number of dimensions of which does not exceed the greatest exponent of the power of the elementary divisors of the matrix $D - \lambda E$ belonging to the distinct E_{s_h}.

§3. THE INVARIANT MANIFOLDS IN THE CASE OF COMPLEX LATENT ROOTS

Assume now that there are complex roots among the latent roots of the matrix D. Since we have limited ourselves to the consideration of matrices with real elements, the complex latent roots must necessarily be conjugate pairs. All results adduced in §2 are transferable to this case in complex space, but for real space corrections must be introduced.

To each pair of conjugate latent roots there corresponds a pair of conjugate complex latent vectors, determining a real invariant two-dimensional plane. Thus in the case in which all the latent roots are distinct, to the invariant directions (for the real latent roots) such invariant planes are added in the role of fundamental elements from which all the invariant manifolds are constructable.

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[1] Several independent axial manifolds can be taken from each E_{s_h}.

In the second case (latent roots multiple, elementary divisors relatively prime), to the two conjugate latent roots $\rho_k \exp(\pm i\phi_k)$ of multiplicity r_k there correspond invariant manifolds of two, four and so forth, up to $2r_k$ dimensions, located one within another. These manifolds may be constructed by means of a system of linearly independent, fundamental vectors (real), $\mathbf{f}_1^k, \mathbf{f}_2^k, \cdots, \mathbf{f}_{2r_k}^k$, satisfying the equations

$$\begin{aligned} D\mathbf{f}_1^k &= \rho_k(\cos\phi_k \mathbf{f}_1^k - \sin\phi_k \mathbf{f}_2^k), & D\mathbf{f}_{2\chi-1} &= \rho_k(\cos\phi_k \mathbf{f}_{2\chi-1}^k - \sin\phi_k \mathbf{f}_{2\chi}^k) + \mathbf{f}_{2\chi-3}^k \\ D\mathbf{f}_2^k &= \rho_k(\sin\phi_k \mathbf{f}_1^k + \cos\phi_k \mathbf{f}_2^k), & D\mathbf{f}_{2\chi} &= \rho_k(\sin\phi_k \mathbf{f}_{2\chi-1}^k + \cos\phi_k \mathbf{f}_{2\chi}^k) + \mathbf{f}_{2\chi-2}^k \end{aligned} \tag{3.1}$$

$$\chi = 2, 3, \cdots, r_k$$

Invariant manifolds of this kind, constructed upon two complex axial manifolds, we shall call imaginary-axial (мнимо-осевое) invariant manifolds, and the system of vectors $\mathbf{f}_1^k, \mathbf{f}_2^k, \cdots, \mathbf{f}_{2r_k}^k$ the fundamental basis in E_{2r_k}.

The structure of the fundamental invariant manifolds that correspond to the complex latent roots in the third case considered in § 2 will change accordingly. We shall not dwell in detail upon this.

4. CERTAIN PROPERTIES OF THE SIMPLEST INVARIANT MANIFOLDS

In the following we shall consider vectors obtained from a certain initial vector $\mathbf{q}_0$ by successive transformations effected by the matrix D, i.e., the vectors

$$\mathbf{q}_k = D^k \mathbf{q}_0, \qquad k = 1, 2, 3, \cdots \tag{4.1}$$

Assume first that the vector $\underset{\sim}{q}_0$ lies in a certain χ-dimensional axial invariant manifold E_χ corresponding to the latent root λ_i. Let $\underset{\sim}{q}_0$ be analyzed according to the vectors $\underset{\sim}{f}_1{}^i, \cdots, \underset{\sim}{f}_\chi{}^i$ forming the fundamental basis of E_χ:

$$\underset{\sim}{q}_0 = c_1\underset{\sim}{f}_1{}^i + c_2\underset{\sim}{f}_2{}^i + \cdots + c_\chi\underset{\sim}{f}_\chi{}^i \tag{4.2}$$

Here it may be assumed that $c_\chi \neq 0$, i.e., that $\underset{\sim}{q}_0$ cannot be placed in an axial manifold of dimension less than χ in number. Then on the basis of equality (2.1) there is readily obtained:

$$\underset{\sim}{q}_p = \lambda_i{}^p(c_1\underset{\sim}{f}_1{}^i + c_2\underset{\sim}{f}_2{}^i + \cdots + c_\chi\underset{\sim}{f}_\chi{}^i) + \lambda_i^{p-1}\binom{p}{1}(c_2\underset{\sim}{f}_2{}^i + c_3\underset{\sim}{f}_3{}^i + \cdots + c_\chi\underset{\sim}{f}_\chi{}^i)$$
$$+ \lambda_i^{p-2}\binom{p}{2}(c_3\underset{\sim}{f}_3{}^i + \cdots + c_\chi\underset{\sim}{f}_\chi{}^i) + \cdots + \lambda_i^{p-\chi+1}\binom{p}{\chi-1}c_\chi\underset{\sim}{f}_\chi{}^i \tag{4.3}$$

From this equality we see that with increasing p,

$$\lim \frac{|\underset{\sim}{q}_p - \lambda_i^{p-\chi+1}\binom{p}{\chi-1}c_\chi\underset{\sim}{f}_\chi{}^i|}{|\underset{\sim}{q}_p|} = 0 \tag{4.4}$$

and accordingly the vector $\underset{\sim}{q}_p$ will approach $\underset{\sim}{f}_\chi{}^i$ in direction.

In the simplest case, when $\underset{\sim}{q}_0$ is collinear with the latent vector ($\chi = 1$), equation (4.3) takes the form

$$\underset{\sim}{q}_p = \lambda_i{}^p\underset{\sim}{q}_0 \tag{4.5}$$

It is known[1] that every vector $\underset{\sim}{q}_0$ belonging to E_χ satisfies the equation

$$(D - \lambda_i)^\chi\underset{\sim}{q}_0 = 0 \tag{4.6}$$

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[1] See, e.g., [6], p. 179.

i.e., that the vectors $\mathbf{q}_0, \mathbf{q}_1, \cdots, \mathbf{q}_\chi$ are connected by a linear dependence

$$\mathbf{q}_\chi - \chi\lambda_i\mathbf{q}_{\chi-1} + \binom{\chi}{2}\lambda_i^{2}\mathbf{q}_{\chi-2}\cdots + (-1)^{\chi}\lambda_i^{\chi}\mathbf{q}_0 = 0 \qquad (4.7)$$

whence we obtain

$$\lambda_i^{r}[\mathbf{q}_0\mathbf{q}_1\cdots\mathbf{q}_{r-1}\mathbf{q}_{r+1}\cdots\mathbf{q}_\chi] = \binom{\chi}{r}[\mathbf{q}_1\mathbf{q}_2\cdots\mathbf{q}_\chi], \quad (r = 1, 2, \cdots, \chi) \qquad (4.8)$$

or

$$(r+1)\lambda_i[\mathbf{q}_0\mathbf{q}_1\cdots\mathbf{q}_{r-1}\mathbf{q}_r\mathbf{q}_{r+2}\cdots\mathbf{q}_\chi] = (\chi-r)[\mathbf{q}_0\mathbf{q}_1\cdots\mathbf{q}_{r-1}\mathbf{q}_{r+1}\mathbf{q}_{r+2}\cdots\mathbf{q}_\chi]$$
$$r = 0, 1, \cdots, \chi - 1 \qquad (4.9)$$

Here, as in what follows, $[\mathbf{q}_1\mathbf{q}_2 \cdots \mathbf{q}_\chi]$ designates the polyvector constructed upon the vectors $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_\chi$. [See Note 1 at end of translation.]

We now adduce analogous formulae for imaginary-axial invariant manifolds. Let us consider first the simplest case, in which $\mathbf{q}_0$ lies in the two-dimensional imaginary-axial manifold that corresponds to the latent roots $\rho_k \exp(\pm i\phi_k)$.

$\mathbf{q}_0$ may then be analysed in terms of the base vectors $\mathbf{f}_1^{k}$ and $\mathbf{f}_2^{k}$:

$$\mathbf{q}_0 = c_1\mathbf{f}_1^{k} + c_2\mathbf{f}_2^{k} = A(\cos\alpha\,\mathbf{f}_1^{k} + \sin\alpha\,\mathbf{f}_2^{k})$$
$$\left(A = \sqrt{c_1^{2} + c_2^{2}}, \qquad \alpha = \arctan\frac{c_2}{c_1}\right) \qquad (4.10)$$

From equations (3.1) we obtain

$$D^p \underset{\sim}{q}_o = A\rho_k{}^p[\cos(\alpha + p\phi_k)\underset{\sim}{f}_1{}^k + \sin(\alpha + p\phi_k)\underset{\sim}{f}_2{}^k] \qquad (4.11)$$

From (4.11), or directly from (3.1) and (4.10) it follows that

$$(D^2 - 2\rho_k \cos \phi_k D + \rho_k{}^2)\underset{\sim}{q}_o = 0 \qquad (4.12)$$

Whence it emerges that

$$[\underset{\sim}{q}_1\underset{\sim}{q}_2] = \rho_k{}^2[\underset{\sim}{q}_o\underset{\sim}{q}_1], \qquad [\underset{\sim}{q}_o\underset{\sim}{q}_2] = 2\rho_k \cos \phi_k[\underset{\sim}{q}_o\underset{\sim}{q}_1] \qquad (4.13)$$

Let us now pass to the general case. Let $\underset{\sim}{q}_o$ belong to the imaginary-axial invariant manifold $E_{2\chi}$ that corresponds to the latent roots $\rho_k \exp(\pm i\phi_k)$, and let the resolution of $\underset{\sim}{q}_o$, in terms of the vectors forming the fundamental basis in $E_{2\chi}$ have the form (we drop the upper indices k):

$$\begin{aligned}\underset{\sim}{q}_o = {} & A_1(\cos \alpha_1\underset{\sim}{f}_1 + \sin \alpha_1\underset{\sim}{f}_2) + A_2(\cos \alpha_2\underset{\sim}{f}_3 + \sin \alpha_2\underset{\sim}{f}_4) + \cdots + \\ & + A_\chi(\cos \alpha_\chi\underset{\sim}{f}_{2\chi-1} + \sin \alpha_\chi\underset{\sim}{f}_{2\chi})\end{aligned} \qquad (4.14)$$

Then on the basis of (3.1) we have

$$\begin{aligned}\underset{\sim}{q}_p = {} & \rho_k{}^p \sum_{j=1}^{\chi} A_j[\cos(\alpha_j + p\phi_k)\underset{\sim}{f}_{2j-1} + \sin(\alpha_j + p\phi_k)\underset{\sim}{f}_{2j}] + \\ & + \binom{p}{1}\rho_k^{p-1} \sum_{j=1}^{\chi-1} A_{j+1}[\cos(\alpha_{j+1} + \{p-1\}\phi_k)\underset{\sim}{f}_{2j-3} + \sin(\alpha_{j+1} + \{p-1\}\phi_k)\underset{\sim}{f}_{2j-2}] + \cdots + \\ & + \binom{p}{2\chi-1}A_\chi\rho_k^{p-\chi+1}[\cos(\alpha_\chi + \{p-\chi+1\}\phi_k)\underset{\sim}{f}_1 + \sin(\alpha_\chi + \{p-\chi+1\}\phi_k)\underset{\sim}{f}_2]\end{aligned} \qquad (4.15)$$

Equation (4.12) may here be replaced by the equation

$$(D^2 - 2\rho_k \cos \phi_k D + \rho_k{}^2)^\chi \underset{\sim}{q}_o = 0 \qquad (4.16)$$

which expresses the sought relation of linear dependence between the vectors $\underset{\sim}{q}_o, \underset{\sim}{q}_1, \cdots, \underset{\sim}{q}_{2\chi}$. From (4.16) it follows that

$$[\underset{\sim}{q}_o \underset{\sim}{q}_1 \cdots \underset{\sim}{q}_{r-1} \underset{\sim}{q}_{r+1} \cdots \underset{\sim}{q}_{2\chi}] = c_r \rho^{2\chi - r} [\underset{\sim}{q}_o \underset{\sim}{q}_1 \cdots \underset{\sim}{q}_{2\chi-1}] \qquad (4.17)$$

where

$$\begin{aligned} c_{2m} &= \sum_i \binom{\chi}{m-i} \binom{\chi-m+i}{2i} 2^{2i} \cos^{2i+1}\phi_k \\ & \qquad\qquad\qquad\qquad (m = 0, 1, \cdots, \chi - 1) \qquad (4.18) \\ c_{2m+1} &= \sum_i \binom{\chi}{m-i} \binom{\chi-m+i}{2i+1} 2^{2i+1} \cos^{2i+1}\phi_k \end{aligned}$$

the summation proceeding, according to i, from 0 to min $(m, \chi - m)$ in the first sum, and to min $(m, \chi - m - 1)$ in the second.

In particular

$$[\underset{\sim}{q}_1 \underset{\sim}{q}_2 \cdots \underset{\sim}{q}_{2\chi}] = \rho_k^{2\chi} [\underset{\sim}{q}_o \underset{\sim}{q}_1 \cdots \underset{\sim}{q}_{2\chi-1}] \qquad (4.19)$$

From formulae (4.18) it is readily seen that $c_r = c_{2\chi - r}$.

§5. THE METHOD OF ITERATION FOR THE DETERMINATION OF THE DOMINANT LATENT ROOT

Let us now designate by $\lambda_1, \lambda_2, \cdots, \lambda_m$ the distinct latent roots of the given matrix $D = \{d_{ik}\}$; further, $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m|$; and by $s_1, s_2, \cdots, s_m$ respectively their multiplicity ($\sum_i s_i = n$). Let the number of complex latent roots (distinct) be equal to 2t.

To each real latent root, and to each pair of conjugate complex latent roots, there corresponds a proper invariant manifold, as has already been shown in §2 and 3.

No two of these manifolds have common directions, and all of them together determine an n-dimensional space, so that an arbitrary vector $\underset{\sim}{a}$ may be resolved into components

$$\underset{\sim}{a} = \underset{\sim}{a}_1 + \underset{\sim}{a}_2 + \cdots + \underset{\sim}{a}_m \tag{5.1}$$

such that each component $\underset{\sim}{a}_i$ belongs to a certain axial invariant manifold $E_{r_i} (r_i \leq s_i)$, the latter corresponding to the latent root λ_i. If a limitation to real space be made, for the complex conjugate latent roots λ_i and $\lambda_{i+1} = \overline{\lambda}_i$, $\underset{\sim}{a}_i$ must be considered as belonging to the corresponding imaginary-axial invariant manifold $E_{2r_i} (r_i \leq s_i)$, whereas $\underset{\sim}{a}_{i+1} = 0$. Depending upon the selection of the vector $\underset{\sim}{a}$, separate summands of sum (5.1) may reduce to zero.

Transforming the vector $\underset{\sim}{a}$ p times by means of the matrix D, we shall obtain

$$D^p\underset{\sim}{a} = D^p\underset{\sim}{a}_1 + D^p\underset{\sim}{a}_2 + \cdots + D^p\underset{\sim}{a}_m \tag{5.2}$$

Each summand in this sum can be expressed in terms of the basic vectors of the corresponding invariant manifolds in accordance with formulae (4.3) and (4.15). From these formulae it follows directly that the ratio $\dfrac{|D^p\underset{\sim}{a}_i|}{|\lambda_1|^p}$ will tend to zero if $|\lambda_i| < |\lambda_1|$, and will grow — or, in the simplest case ($\underset{\sim}{a}_i$ coincides with the invariant direction), remain constant with $|\lambda_i| = |\lambda_1|$. [See Note 2 at end of translation.]

Therefore no matter what the assigned degree of accuracy may be, given a sufficiently large p it is possible to consider the vector $D^p \underset{\sim}{a}$ as lying in the invariant manifold formed by the fundamental axial and imaginary-axial manifolds that correspond to the dominant latent roots, if only the vector $\underset{\sim}{a}$ contains a counterpart in this manifold (fundamental theorem).

If in sum (5.1) the first summand different from zero be $\underset{\sim}{a}_k$, then λ_1 should be replaced by λ_k everywhere above. It is nevertheless practically always possible to consider $\underset{\sim}{a}_1 \neq 0$. Let alone the fact that the probability of the absence of a component of the arbitrarily selected initial vector in some determinate invariant manifold or another is equal to zero, it should be pointed out that inasmuch as the computations are being conducted to a limited number of decimal places, the component $\underset{\sim}{a}_1$ will be present if only because of the rounding error — and this will be so whether or not the attempt be made to select vector $\underset{\sim}{a}$ such that it is known to contain no component[1] $\underset{\sim}{a}_1$. This circumstance, as will be shown in what follows, considerably complicates the determination of the latent roots whose moduli are less than that of λ_1.

The method for the approximate determination of the dominant latent roots by successive transformations of an arbitrary vector $\underset{\sim}{a}$ by means of the matrix D is based upon the fundamental theorem. For several of the simplest cases this method has been investigated in the works of Mises and of Duncan cited above, and — with a more general view — in the work of Aitken. The results obtained in §6-10 of the present work are partially contained in Aitken's work.

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[1] See, e.g., [1], p. 157.

6. THE GENERAL CASE

Let us consider the sequence of vectors

$$\underset{\sim}{a},\ D\underset{\sim}{a},\ D^2\underset{\sim}{a},\ \cdots,\ D^p\underset{\sim}{a},\ \cdots \tag{6.1}$$

where $\underset{\sim}{a}$ is an arbitrary vector. If r be the maximum number of linearly independent vectors in this sequence, the vectors $\underset{\sim}{a}, D\underset{\sim}{a}, \cdots, D\underset{\sim}{a}^{r-1}$ will be independent, and having determined the coefficients of the relation of linear dependence holding among the first r + 1 vectors of the sequence (6.1)

$$\gamma_0 D^r\underset{\sim}{a} + \gamma_1 D^{r-1}\underset{\sim}{a} + \cdots + \gamma_{r-1} D\underset{\sim}{a} + \gamma_r\underset{\sim}{a} = 0 \tag{6.2}$$

it is possible to write the characteristic equation

$$\gamma_0\lambda^r + \gamma_1\lambda^{r-1} + \cdots + \gamma_{r-1}\lambda + \gamma_r = 0 \tag{6.3}$$

for the matrix defining the transformation induced by transformation L (with matrix D) in the minimal invariant manifold containing the vector $\underset{\sim}{a}$. This is the substance of the method elaborated by A. N. Krylov and N. N. Luzin for composing the characteristic equation.

From the preceding paragraph it follows that, to the assigned degree of accuracy, the number (which we shall designate by m) of successive linearly independent vectors in sequence (6.1), beginning with $D^p\underset{\sim}{a}$, will diminish with the growth of p. If the coefficients of the linear dependence holding among the vectors $D^p\underset{\sim}{a}, \cdots, D^{p+m}\underset{\sim}{a}$ be determined:

$$c_0 D^{p+m}\underset{\sim}{a} + c_1 D^{p+m-1}\underset{\sim}{a} + \cdots + c_{m-1}D^{p+1}\underset{\sim}{a} + c_m D^p\underset{\sim}{a} = 0 \tag{6.4}$$

then with a sufficiently large p all roots of the equation

$$c_0\lambda^m + c_1\lambda^{m-1} + \cdots + c_{m-1}\lambda + c_m = 0 \qquad (6.5)$$

will be equal in modulus and will coincide with the roots of equation (6.3) that are of greatest modulus.

Taking p sufficiently large, the degree of equation (6.5) may be reduced to the number of dominant latent roots having elementary divisors of highest degree.

Actually, it follows from formula (4.4) that among the vectors $D^p\underset{\sim}{a}_i$ corresponding to the latent roots with equal moduli, with increasing p that particular vector increases most rapidly for which the degree of the corresponding elementary divisor is greatest.

Thus having taken an arbitrary vector $\underset{\sim}{a}$, after a sufficiently large number of transformations of it by means of matrix D, by the aforesaid method an equation (6.5) can be obtained, the roots of which will be the dominant latent roots.

The question, how numerous must the transformations be? is difficult to decide without knowing at least approximately the magnitude of the ratios of the latent roots and the multiplicity of the corresponding elementary divisors. Having completed a transformation and found the coefficients of the relation of linear dependence between the vectors $D^p\underset{\sim}{a}, \cdots, D^{p+m}\underset{\sim}{a}$, the question whether p is sufficiently large can be answered by testing whether the moduli of the roots of equation (6.5) are equal. The necessary and sufficient conditions to be imposed upon the coefficients of the equation in order that there be equality of moduli

of the roots may be met by the introduction of a new unknown $\mu = |c_0 : c_m|^{\frac{1}{m}} \lambda$ and by the application of the criterion of Cohn for the distribution of the roots on the unit circumference[9]. For m = 2, 3, 4, 5, a summary of these conditions is set forth below; these may by elementary methods be obtained readily for the given most simple cases (for simplicity c_0 has been taken equal to 1). In contradistirction to the general criterion of Cohn, a preliminary removal of the small roots is not here presupposed.

1) $m = 2$. Equation $\lambda^2 + c_1 \lambda + c_2 = 0$.

Condition for the equality of the moduli of the roots:

$$c_1^4 \leq 4c_2c_1^2$$

2) $m = 3$. Equation $\lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3 = 0$.

Conditions for the equality of the moduli of the roots:

$$c_2^3 = c_3c_1^3 , \qquad |c_1^2 - c_2| \leq 2|c_2|$$

3) $m = 4$. Equation $\lambda^4 + c_1 \lambda^3 + c_2 \lambda^2 + c_3 \lambda + c_4 = 0$.

Conditions for the equality of the moduli of the roots:

$$c_3 = c_1\sqrt{c_4}, \; c_1^3 \leq 16c_3, \; 2|c_1| - \frac{2}{\sqrt{c_4}} \leq c_2 \leq \frac{c_1^2}{4} + \frac{2}{\sqrt{c_4}} \quad \text{for} \quad c_4 > 0$$

or

$$c_2 = 0, \; c_3 = -c_1\sqrt{-c_4}, \; c_1^2 - 4\sqrt{-c_4} \leq 0 \quad \text{for} \quad c_4 \leq 0$$

4) n = 5. Equation $\;^5 + c_1 \;^4 + c_2 \;^3 + c_3 \;^2 + c_4 \; + c_5 = 0$ Put $c_5 = r^5 \neq 0$. Then the conditions for the equality of the moduli of the roots take the form

$$c_3 = rc_2, \; c_4 = r^3 c_1, \; \left|\frac{c_1}{r} - 1\right| \leq 4, \; 2\left|\frac{c_1}{r} - 1\right| \leq 3 + \frac{c_2}{r^2} - \frac{c_1}{r} \leq 4 + \frac{1}{4}\left(\frac{c_1}{r} - 1\right)^2$$

If a check reveals the condition for the equality of the moduli of the roots for equation (6.5) to have been satisfied, the roots may then easily be determined. First the real roots, equal to $\pm|c_n|^{\frac{1}{m}}$, may be removed. After this separation of the real, the complex roots can then be determined by the methods ordinarily applied for the solution of equations by the method of Graeffe.

Following the determination of the latent roots, the corresponding invariant manifolds are found. Since the vectors belonging to the simplest invariant manifolds satisfy equations of form (4.6) or (4.12), constituting determinate linear combinations of the vectors $D^p\underset{\sim}{a}, \cdots, D^{p+m}\underset{\sim}{a}$, it is possible to obtain the vectors forming the basis of the simplest invariant manifold corresponding to one or another latent root, or, more correctly, of that part of the manifold that is contained in the invariant manifold defined by the vectors $D^p\underset{\sim}{a}, \cdots, D^{p+m}\underset{\sim}{a}$. Actually, letting λ^* be some root of equation (6.5) of multiplicity k, the vectors

$$(D - \lambda^*)^k D^{p+1}\underset{\sim}{a}, \qquad i = 0, 1, \cdots, m - k$$

will not have components belonging to the invariant manifolds corresponding to λ^*. Of the system of vectors obtained, the components that belong to the invariant manifolds corresponding to whichsoever other latent roots of those obtained may be annihilated in like manner.

7. SOME FREQUENT CASES

In practice cases in which a significant number of latent roots coincide in modulus are rarely encountered, nor are cases in which elementary divisors of high degree are present. A more detailed investigation of the process of determining the latent roots of the invariant manifolds is made below for the simplest cases. This will be sufficient in the surpassing majority of practical problems.

1. Simplest will be the case where λ_1 is a simple latent root and $|\lambda_2| \neq |\lambda_1|$. In this case for sufficiently large p the direction of the vector $D^p\underset{\sim}{a}$ will coincide with the invariant direction corresponding to λ_1, and accordingly, beginning with a certain p, the direction of the vectors $D^{p+i}\underset{\sim}{a}$ will coincide the one with its predecessor — i.e., their coordinates will be proportional. This proportionality of coordinates serves also as an indicator that the direction of the vectors of our sequence $D^{p+i}\underset{\sim}{a}$ coincides, within the limits of accuracy of the computation, with the invariant direction of the matrix D.

The latent root corresponding to this invariant direction will be equal to the ratio of corresponding coordinates of the vectors $D^{p+1}\underset{\sim}{a}$ and $D^p\underset{\sim}{a}$.

For computational convenience, in place of the sequence of vectors $D^p\underset{\sim}{a}$ $(p = 1, 2, \cdots)$, a sequence of vectors $\underset{\sim}{b}_p$ could be considered, defined by the recurrence formula $\underset{\sim}{b}_{p+1} = \mu_p D\underset{\sim}{b}_p$, $\underset{\sim}{b}_o = \underset{\sim}{a}$. Here μ_p are arbitrary numbers selected in answer to the demand for simplification of the computation. With the growth of p, $\underset{\sim}{b}_p$ will approximate $\underset{\sim}{f}_1$ in direction; the ratio of corresponding coordinates of the vectors $\underset{\sim}{b}_{p+1}$ and $\underset{\sim}{b}_p$ will approach $\mu_p \lambda_1$.

The introduction of auxiliary multipliers should be recommended in all the cases following as well, but we shall not do this out of concern for simplicity of exposition.

We shall obtain just such a picture if λ_i is a multiple latent root of multiplicity r, but the corresponding elementary divisors are all of the first degree (here as before $|\lambda_2| \neq |\lambda_1|$). The invariant directions obtained in the manner indicated above will here ordinarily be distinct for distinct initial vectors $\underset{\sim}{a}$; they will, however, belong to one and the same r-dimensional manifold corresponding to λ_1, in which any direction is invariant (v. § 2, case 3).

If we find the invariant direction, using any initial vector whatsoever, but only one such, then these two cases will be indistinguishable.

Let us now assume that with a real latent root λ_1 there correspond the elementary divisors $(\lambda - \lambda_1)^{\chi_1}, \dots, (\lambda - \lambda_1)^{\chi_i}$, $\chi_1 \leq \chi_2 \leq \dots \leq \chi_i$.

Then, as was pointed out in § 2, the component of an arbitrary vector $\underset{\sim}{a}$ with respect to an invariant, s_1-dimensional ($s_1 = \chi_1 + \chi_2 + \dots + \chi_i$) manifold E_{s_1} corresponding to λ_1, if only it does not equal zero, will always belong to a certain axial invariant manifold E_k lying within E_{s_1}, and $k \leq \chi_i$. If only $|\lambda_2| \neq |\lambda_1|$, then on the strength of the fundamental theorem it may be considered that the vector $\underset{\sim}{q}_0 = D^p\underset{\sim}{a}$ for sufficiently large p, and within the limits of admissible accuracy, belongs to E_k.

The equality of the moduli of the roots of equation (6.5) betokens, as has been indicated above, the fulfilment of the condition of the condition of the fundamental theorem and the fact that the process of computing the vectors $D^m\underset{\sim}{a}$ may be terminated. In the case in hand equation (6.5) must take the form

$$(\lambda - \lambda_1)^k = 0 \qquad (7.1)$$

which may be easily established after finding the maximum number k of linearly independent vectors in the sequence $\underset{\sim}{q}_o = D^p\underset{\sim}{a}$, $\underset{\sim}{q}_i = D^i\underset{\sim}{q}_o$, $(i = 1, 2, \dots)$ and obtaining the equation of linear dependence (6.4).

Practically, here as in the rest of the simplest cases, it is not necessary to find all the coefficients of linear dependence (6.4). Equations (4.8) and (4.9) are consequences of equation (4.7), and the two former may serve for the determination of λ_1 on the supposition that equation (6.5) reduces to (7.1). Having determined λ_1 by means of any two of these equations (only two or three of the coordinates of the polyvectors are usually satisfied in the determination), one can judge the correctness of the assumption made by the coincidence of the roots obtained. However this coincidence cannot, strictly speaking, serve as a precise criterion whether equation (6.4) has the form (4.7), and a definitive verification of the value of λ_1 that is found is to be made simultaneously with the verification of the invariant direction, discussed below.

Let it now be established that $\underset{\sim}{q}_o$ belongs to a certain invariant axial manifold E_k corresponding to the latent root λ_1.

The vectors $\underset{\sim}{q}_1, \underset{\sim}{q}_2, \cdots, \underset{\sim}{q}_k$ may serve as a basis for this manifold, but they will not form a fundamental basis.

Let us now construct the groups of vectors $\underset{\sim}{q}_i^{\,r}$ according to the following formulae:

$$\begin{aligned} &\underset{\sim}{q}_i^{\,o} = \underset{\sim}{q}_i, && (i = 1, 2, \cdots, k) \\ &\underset{\sim}{q}_i^{\,r} = \underset{\sim}{q}_{i+1}^{\,r-1} - \lambda_1 \underset{\sim}{q}_i^{\,r-1}, && \begin{pmatrix} i = 1, 2, \cdots, k - r \\ r = 1, 2, \cdots, k - 1 \end{pmatrix} \end{aligned} \qquad (7.2)$$

Since the vectors $\underset{\sim}{q}_i^{\,o}$ satisfy the equation $(D - \lambda_i)^k \underset{\sim}{q}^{\,o} = 0$, the vectors $\underset{\sim}{q}_i^{\,r}$ will satisfy the equation

$$(D - \lambda_1)^{k-r} \underset{\sim}{q}_i^{\,r} = 0$$

Moreover, the vectors $\underset{\sim}{q}_i^{\,r}$ for fixed r will be linearly independent on the strength of the linear independence of the vectors $\underset{\sim}{q}_i$ and may accordingly be taken as a basis of the axial manifold E_{k-r}. In particular, the direction of vector $\underset{\sim}{q}_1^{\,k-1}$ will coincide with that of the latent vector corresponding to λ_1. The vectors $\underset{\sim}{q}_1^{\,k-1}, \underset{\sim}{q}_1^{\,k-2}, \cdots, \underset{\sim}{q}_1^{\,1}, \underset{\sim}{q}_1^{\,o}$ form a basis in E_k that is fundamental. In fact, they are linearly independent on the strength of the independence of $\underset{\sim}{q}_1, \cdots, \underset{\sim}{q}_k$ and satisfy relations (2.1), which follows from (7.2). The computation of $D^{k-1}\underset{\sim}{q}_1^{\,k-1}$ gives conclusive verification of the accuracy of the computations and of the justice of the application of formulae (4.8) and (4.9) since, obviously, only in the case of the correctness of all assumptions made will we obtain $D^{k-1}\underset{\sim}{q}_1^{\,k-1} = \lambda_1 \underset{\sim}{q}_1^{\,k-1}$.

2. Let us pass now to the next case, where $\lambda_1 = \bar{\lambda}_2$ and $|\lambda_3| \neq |\lambda_1|$. Let the dominant latent roots be the complex quantities $\lambda_1 = \rho e^{i\phi}$ and $\lambda_2 = \rho e^{-i\phi}$, the corresponding elementary divisors being all of the first degree, and $|\lambda_3| \neq \rho$.

Then, on the strength of the fundamental theorem, with sufficiently large p the vector $D^p \underset{\sim}{a} = \underset{\sim}{q}_o$ will belong to the two-dimensional imaginary-axial invariant plane that corresponds to our latent roots, and equations (4.13) will be satisfied accordingly. From them ρ and ϕ may be determined. For basic vectors there may be taken

$$\underset{\sim}{f}_1 = -\rho \cos \phi \underset{\sim}{q}_o + \underset{\sim}{q}_1 \quad \text{and} \quad \underset{\sim}{f}_2 = \rho \sin \phi \underset{\sim}{q}_o$$

It may be readily verified that the first two equations of (3.1) will be satisfied.

If the complex latent roots are multiple with elementary divisors $(\lambda - \lambda_1)^{\chi_1}, \cdots, (\lambda - \lambda_1)^{\chi_m}$,
$(\lambda - \lambda_2)^{\chi_1}, \cdots, (\lambda - \lambda_2)^{\chi_m}$, the vector $D^p \underset{\sim}{a} = \underset{\sim}{q}_o$ with sufficiently large p will belong to a certain imaginary-axial invariant manifold $E_{2\chi}$, where $\chi \leq \chi_m$ depends on the choice of $\underset{\sim}{a}$ and, accordingly, equations (4.17) will be satisfied. The first two, or the first and last of them, give the possibility of readily determining ρ and ϕ. For control it is better to make use of at least three of the equations.

Vectors $\underset{\sim}{q}_1, \underset{\sim}{q}_2, \cdots, \underset{\sim}{q}_{2\chi}$ are linearly independent and can serve as basis for $E_{2\chi}$. Let us construct now the vectors

$$\begin{aligned} &\underset{\sim}{q}_i{}^{o} = \underset{\sim}{q}_i && (i = 1, 2, \cdots, 2\chi) \\ &\underset{\sim}{q}_i{}^{r} = (D^2 - 2\rho \cos \phi D + \rho^2)\underset{\sim}{q}_i^{r-1} = \underset{\sim}{q}_{i+2}^{r-1} - 2\rho \cos \phi \underset{\sim}{q}_{i+1}^{r-1} + \rho^2 \underset{\sim}{q}_i^{r-1} && (7.3) \\ &\qquad\qquad \begin{pmatrix} r = 1, 2, \cdots, \chi - 1 \\ i = 1, 2, \cdots, 2(\chi - r) \end{pmatrix} \end{aligned}$$

The vectors $\underset{\sim}{q}_i{}^r[i = 1, 2, \cdots, 2(\chi - r)]$ can serve as a basis for the imaginary-axial invariant manifold $E_{2(\chi-r)}$ contained in $E_{2\chi}$ since they are linearly independent and satisfy the equation $(D^2 - 2\rho \cos \phi D + \rho^2)^{\chi-r}\, \underset{\sim}{q}_i = 0$ which follows from their definition. The system of vectors $\underset{\sim}{q}_1{}^r$, $\underset{\sim}{q}_2{}^r$, $(r = \chi, \chi - 1, \cdots, 2, 1)$, despite that it will not be a fundamental basis — since equations (3.1) will not be satisfied — will nevertheless possess, as may be readily confirmed, the most important property of a fundamental basis, namely, that the first 2k vectors of this system form the basis of the imaginary-axial manifold E_{2k} contained in $E_{2\chi}$.

3. Let us suppose that the matrix D has several distinct latent roots, each of these having the same modulus, and this the greatest. If a limitation to real latent roots only be made, a case is here possible in which $\lambda_1 = -\lambda_2$. Let the corresponding elementary divisors be

$$(\lambda - \lambda_1)^{\chi_1}, \cdots, (\lambda - \lambda_1)^{\chi_m} \text{ and } (\lambda + \lambda_1)^{h_1}, \cdots, (\lambda + \lambda_1)^{h_k}$$

$$(\chi_1 \le \chi_2 \le \cdots \le \chi_m;\ h_1 \le h_2 \le \cdots \le h_k)$$

We introduce the designations: $\chi_1 + \chi_2 + \cdots + \chi_m = s_1$ and $h_1 + h_2 + \cdots + h_k = s_2$. Then the components $\underset{\sim}{a}_1$ and $\underset{\sim}{a}_2$ of the arbitrary vector a in the corresponding invariant manifolds E_{s_1} and E_{s_2} will belong to certain invariant axial manifolds E_{χ_1} and E_{χ_2}, where $\chi_1 \leq \chi_m$, $\chi_2 \leq h_k$. For definiteness let $\chi_1 \geq \chi_2$. On the strength of the fundamental theorem the vector $\underset{\sim}{q}_o = D^p \underset{\sim}{a}$ with sufficiently large p will satisfy the equation

$$(D - \lambda_1)^{\chi_1}(D + \lambda_1)^{\chi_2}\underset{\sim}{q}_o = 0 \tag{7.4}$$

and accordingly

$$(-1)^{\chi_1}\lambda_1^{\chi_1+\chi_2}[\underset{\sim}{q}_o\underset{\sim}{q}_1 \cdots \underset{\sim}{q}_{\chi_1+\chi_2-1}] = [\underset{\sim}{q}_1\underset{\sim}{q}_2 \cdots \underset{\sim}{q}_{\chi_1+\chi_2}] \tag{7.5}$$

However this equality as well as others analogous that may be obtained from (7.4) are little suitable for the determination of λ_1 and the fundamental vectors.

From equation (7.4) it follows that $\underset{\sim}{q}_o$ and $\underset{\sim}{q}_1$ satisfy the equation

$$(D^2 - \lambda_1{}^2)^{\chi_1}\underset{\sim}{q} = 0 \tag{7.6}$$

whence it issues that

$$[\underset{\sim}{q}_o\underset{\sim}{q}_2\cdots\underset{\sim}{q}_{2r-2}\underset{\sim}{q}_{2r+2} \cdots \underset{\sim}{q}_{2\chi_1}] = \binom{\chi_1}{r}\lambda_1^{2(\chi_1-r)}[\underset{\sim}{q}_o\underset{\sim}{q}_2\cdots\underset{\sim}{q}_{2\chi_1-2}]$$

$$(r = 0, 1, \cdots, \chi_1 - 1) \tag{7.7}$$

$$[\underset{\sim}{q}_1\underset{\sim}{q}_3\cdots\underset{\sim}{q}_{2r-1}\underset{\sim}{q}_{2r+3}\cdots\underset{\sim}{q}_{2\chi_1+1}] = \binom{\chi_1}{r}\lambda_1^{2(\chi_1-r)}[\underset{\sim}{q}_1\underset{\sim}{q}_3\cdots\underset{\sim}{q}_{2\chi_1-1}]$$

and the coordinates of the polyvectors $[\underset{\sim}{q}_o\underset{\sim}{q}_2 \cdots \underset{\sim}{q}_{2\chi_1-2}]$ and $[\underset{\sim}{q}_1\underset{\sim}{q}_3 \cdots \underset{\sim}{q}_{2\chi_1-1}]$ will not be proportional. Equations (7.7) perfectly coincide in essence with equalities (4.8), since they are obtained from (7.6) just as (4.8) is from (4.6). From them λ_1 is readily found. Having determined the number of linearly independent vectors among $\underset{\sim}{q}_1, \underset{\sim}{q}_2, \cdots, \underset{\sim}{q}_{2\chi_1}$, which number must be equated to $\chi_1 + \chi_2$, χ_2 also can be determined. For the determination of the fundamental bases in E_{χ_1} and E_{χ_2} let us proceed in the following manner. Construct the two series of vectors

$$\underset{\sim}{p}_i^o = \underset{\sim}{q}_i^o = \underset{\sim}{q}_i \ (i = 1,2,\cdots,2\chi_1), \ \underset{\sim}{p}_i^j = \underset{\sim}{p}_{i+1}^{j-1} - \lambda_1^{j-1}\underset{\sim}{p}_i$$

$$\begin{pmatrix} j = 1,2,\cdots,\chi_1 \\ i = 1,2,\cdots,2\chi_1 - j \end{pmatrix} \qquad (7.8)$$

$$\underset{\sim}{q}_i^j = \underset{\sim}{q}_{i+1}^{j-1} + \lambda_1^{j-1}\underset{\sim}{q}_i$$

Since

$$(D - \lambda_1)^{\chi_1}\underset{\sim}{x} = 0$$

if $\underset{\sim}{x}$ lies in E_{χ_1}, and $\underset{\sim}{p}_i^{\chi_1} = (D - \lambda_1)^{\chi_1}\underset{\sim}{q}_i$, which follows directly from formulae (7.7), then $\underset{\sim}{p}_i^{\chi_1}$ lies in E_{χ_2}. In the same fashion $\underset{\sim}{q}_i^{\chi_1}$ may easily be shown to lie in E_{χ_1}.

The further determination of the fundamental bases defining E_{χ_1} and E_{χ_2} proceeds just as in case 2 already considered. We find the vectors

$$\underset{\sim}{p}_i^{\chi_1+\chi} = \underset{\sim}{p}_{i+1}^{\chi_1+\chi-1} + \lambda_1\underset{\sim}{p}_i^{\chi_1+\chi-1} \qquad (i = 1, 2, \cdots, \chi_1 - \chi)$$

With $\chi = \chi_2 - 1$ all the vectors $\underset{\sim}{p}_i^{\chi+\chi_1-1}$ will coincide in direction, which will also be the invariant direction belonging to E_{χ_2}. For the fundamental basis the vectors $\underset{\sim}{p}_i^{\chi_1+\chi}$ $(\chi = 0,1,2,\cdots,\chi_2 - 1)$ may be taken. The fundamental basis for E_{χ_2} is determined analogously in accordance with the vectors $\underset{\sim}{q}_i^{\chi_1}$.

It is to be remarked that had the number χ_2 been determined in the manner indicated above, it would have been sufficient to take only $\chi_1 + \chi_2$ of the vectors $\underset{\sim}{q}_i^{\,0}$ and $\underset{\sim}{p}_i^{\,0}$. We would then have obtained χ_2 of the vectors $\underset{\sim}{p}_i^{\chi_1}$ defining E_{χ_2}, and χ_1 of the vectors $\underset{\sim}{q}_i^{\chi_2}$ defining E_{χ_1}.

In conclusion be it remarked that in the simplest case, when $\chi_1 = \chi_2 = 1$, formulae (7.7) will give us

$$\underset{\sim}{q}_2 = \lambda_1^{\,2}\underset{\sim}{q}_0, \quad \underset{\sim}{q}_3 = \lambda_1^{\,2}\underset{\sim}{q}_1 \tag{7.9}$$

The invariant directions corresponding to λ_1 and $\lambda_i = - \lambda_1$ will coincide with $\underset{\sim}{q}_2 + \lambda_1\underset{\sim}{q}_1$ and $\underset{\sim}{q}_2 - \lambda_1\underset{\sim}{q}_1$.

§8. SEVERAL FREQUENT CASES FOR WHICH THE DEGREE OF EQUATION (6.5) DOES NOT EXCEED 4

Of the possible simplest cases, we shall yet consider those for which the degree of equation (6.5) does not exceed four. This is possible, excepting those cases already considered, if

1) $\lambda_1 = \bar{\lambda}_2 = \rho e^{i\phi}, \quad \lambda_3 = \epsilon\rho, \quad |\lambda_4| < \rho,$ where $\epsilon = \pm 1$

2) $\lambda_1 = \bar{\lambda}_2 = \rho e^{i\phi}, \quad \lambda_3 = \rho, \quad \lambda_4 = -\rho \quad |\lambda_5| < \rho$

3) $\lambda_1 = \bar{\lambda}_2 = \rho e^{i\phi_1}, \quad \lambda_3 = \bar{\lambda}_4 = \rho e^{i\phi_2}, \quad |\lambda_5| < \rho$

along with which the elementary divisors corresponding to the indicated latent roots will be of the first degree[1].

On the strength of the fundamental theorem, with sufficiently large p the vector $D^p\underset{\sim}{a} = \underset{\sim}{q}_o$ will satisfy the equation (the enumeration is with respect to the cases above)

$$1)\quad (D^2 - 2\rho\cos\phi + \rho^2)(D - \epsilon\rho)\underset{\sim}{q}_o = 0 \quad ,$$

$$2)\quad (D^2 - 2\rho\cos\phi + \rho^2)(D^2 - \rho^2)\underset{\sim}{q}_o = 0 \qquad (8.1)$$

$$3)\quad (D^2 - 2\rho\cos\phi_1 + \rho^2)(D^2 - 2\rho\cos\phi_2 + \rho^2)\underset{\sim}{q}_o = 0$$

From these equations the fundamental relations permitting the determination of ρ and ϕ may at once be obtained:

$$1)\quad [\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_3] = \epsilon\rho^3[\underset{\sim}{q}_o\underset{\sim}{q}_1\underset{\sim}{q}_2], \quad [\underset{\sim}{q}_o\underset{\sim}{q}_2\underset{\sim}{q}_3] = \epsilon\rho[\underset{\sim}{q}_o\underset{\sim}{q}_1\underset{\sim}{q}_3], \qquad (8.2)$$

$$[\underset{\sim}{q}_o\underset{\sim}{q}_1\underset{\sim}{q}_3] = \rho(\epsilon + 2\cos\phi)[\underset{\sim}{q}_o\underset{\sim}{q}_1\underset{\sim}{q}_2]$$

$$2)\quad [\underset{\sim}{q}_o\underset{\sim}{q}_1\underset{\sim}{q}_3\underset{\sim}{q}_4] = 0 \text{ (but } [\underset{\sim}{q}_o\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_3] \neq 0), \quad [\underset{\sim}{q}_1\underset{\sim}{q}_3\underset{\sim}{q}_4] = \rho^4[\underset{\sim}{q}_o\underset{\sim}{q}_1\underset{\sim}{q}_3], \qquad (8.3)$$

$$[\underset{\sim}{q}_o\underset{\sim}{q}_3\underset{\sim}{q}_4] = \rho^2[\underset{\sim}{q}_o\underset{\sim}{q}_1\underset{\sim}{q}_4], \quad [\underset{\sim}{q}_o\underset{\sim}{q}_1\underset{\sim}{q}_4] = 2\rho\cos\phi[\underset{\sim}{q}_o\underset{\sim}{q}_1\underset{\sim}{q}_3]$$

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[1] Case 3 may be considered as including two possibilities omitted from consideration, to wit: the presence of elementary divisors of the second degree for λ_3 in case 1, or the presence of elementary divisors of the second degree for a single pair of complex latent roots. Actually the corresponding equation for $\underset{\sim}{q}_o$ will coincide with equation (8.1) with $\phi_2 = 0$ or $\phi_2 = \pi$ or $\phi_1 = \phi_2$. The obtaining of these values of ϕ_2 from equations (8.4) will serve as a signal of the presence of elementary divisors of the second degree, in accordance with which corresponding changes in the process of finding the invariant manifolds must be introduced.

$$3)\quad [\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_3\underset{\sim}{q}_4] = \rho^4[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_3], \quad [\underset{\sim}{q}_0\underset{\sim}{q}_2\underset{\sim}{q}_3\underset{\sim}{q}_4] = \rho^2[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_4],$$

$$[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_4] = 2\rho(\cos \phi_1 + \cos \phi_2)[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_3], \qquad (8.4)$$

$$[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_3\underset{\sim}{q}_4] = 2\rho^2(1 + 2 \cos \phi_1 \cos \phi_2)[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_3]$$

A coincidence of values for ρ obtained from the first two of the equations of each group can serve as a control of the correctness of the hypothesis that the conditions of the fundamental theorem have already been fulfilled and that equation (6.4) has in fact the form of one of the equations (8.1).

The determination of the invariant manifolds is here possible by the general method indicated at the end of §6. Thus for example, in case 1 the vector $(D^2 - 2\rho \cos \phi + \rho^2)\underset{\sim}{q}_0$ will give the invariant direction corresponding to λ_3, and the vectors $(D - \lambda_3)\underset{\sim}{q}_0$ and $(D - \lambda_3)\underset{\sim}{q}_1$ will determine the imaginary-axial manifold E_2 corresponding to $\lambda_{1,2}$. Analogous formulae may be readily obtained for the rest of the cases as well.

§9. PRACTICAL APPLICATIONS OF THE METHOD OF ITERATION

In computations the determination of the dominant latent roots of the given matrix D and of the corresponding invariant manifolds are effected in the following manner.

An initial vector $\underset{\sim}{a}$ is selected; this ordinarily has only one coordinate different from zero, or has all coordinates equaling 1. The vectors $D^m\underset{\sim}{a}$ $(m = 1, 2, 3, \cdots)$ are computed[1]. The nature of the change of the vectors of this sequence points to

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[1] Or rather, the vectors $\underset{\sim}{b}_m = \mu_m D\underset{\sim}{b}_{m-1}$; see §7.

one or other case of the presence of distinct roots of greatest modulus. If the vectors $D^m\underset{\sim}{a}$, with a large m, change direction insignificantly, this points to the presence of an elementary divisor of degree higher than the first for latent root λ_1, or to the presence of roots of close modulus. If the methods of § 7 do not give satisfactory results the iterations must be continued for the separation of the unequal roots.

The presence of complex roots (§ 7, case 2) may be revealed by strong oscillations of the directions of the vectors $D^m\underset{\sim}{a}$, marked by oscillations of the signs of these vectors' coordinates. If however the vectors $D^m\underset{\sim}{a}$, $D^{m+2}\underset{\sim}{a}$, $D^{m+4}\underset{\sim}{a}$, ••• slowly change direction, as do vectors $D^{m+1}\underset{\sim}{a}$, $D^{m+3}\underset{\sim}{a}$, ••• , then case 3 of § 7 holds.

If with sufficiently large m one cannot succeed in identifying one of the three possible simplest cases considered in § 7, one is obliged to search for the least number of linearly independent vectors in the sequence $D^m\underset{\sim}{a}$, $D^{m+1}\underset{\sim}{a}$, ••• and to determine the coefficients of the relation of linear dependence (6.4) between them. For this compute here and there a few of the determinants of second, third and higher orders that are the coordinates of the corresponding polyvectors, until a proportionality of the coordinates of the different polyvectors of the first order is obtained. If the number of linearly independent vectors does not exceed four, the formulae of § 8 may be utilized; in the contrary case the general method (§ 6)[1] must be employed.

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[1] See the examples of computations at the end of this work (§ 14).

In conclusion we make one more remark of a practical character. Instead of carrying on the computation with a sequence of vectors

$$\underset{\sim}{a},\ D\underset{\sim}{a},\ D^2\underset{\sim}{a},\ \cdots,\ D^m\underset{\sim}{a} \qquad (9.1)$$

one could consider the sequence of matrices

$$D,\ D^2,\ D^4,\ D^8,\ \cdots \qquad (9.2)$$

by means of which analogous results would be obtained. In the simplest case, in which all the latent roots are real and distinct, in the limit there would be obtained a degenerating matrix the rows of which would be proportional to the coordinates of the latent vector corresponding to the latent root λ_1. At first sight the second method would seem to be preferable in practice, since it secures a more rapid convergence. However this is not entirely true, because to obtain one member of sequence (9.2) one must carry through the same computational work as one would to obtain n members of sequence (9.1).

Since it is important for us to expend the least labor to arrive at the power m necessary in the given case, obviously the first method may prove to be the more advantageous.

For example, if the degeneration indicated above is arrived at, within the limits of computational accuracy, with $m = 16$, the first method would be more advantageous than the second for $n > 4$; if it would be sufficient to take $m = 30$, the first method would be the more advantageous for $n > 6$.

A combination of the computational methods is, however, most expedient. At first find D^2, and perhaps even D^4, and then consider

the sequence

$$\underset{\sim}{a},\ D^2\underset{\sim}{a},\ D^4\underset{\sim}{a},\ \cdots \quad \text{or} \quad \underset{\sim}{a},\ D^4\underset{\sim}{a},\ D^8\underset{\sim}{a},\ D^{16}\underset{\sim}{a},\ \cdots \qquad (9.3)$$

The invariant manifolds of the matrices D and D^2 coincide[1], and the latent roots of D^2 are equal to the square of the latent roots of matrix D. Therefore to obtain results in sequence (9.3) one can take half as many members as in (9.1), which appreciably reduces the quantity of computations.

To assist visualization, a table is subjoined exhibiting the quantity of computations for various n and m by diverse methods. For a unit there has been adopted the quantity of computation necessary to complete one transformation of the vector by means of the matrix D; (obviously these units are not of equal value for differing n).

Computational Method	m = 16			m = 32			m = 64		
	n = 4	n = 6	n = 8	n = 4	n = 6	n = 8	n = 4	n = 6	n = 8
Iteration by D	16	16	16	32	32	32	64	64	64
" " D^2	12	14	16	20	22	24	40	42	44
" " D^4	12	16	20	16	20	24	24	28	32
" " D^8	14	20	26	16	22	28	20	26	32
Succeeding Involution	16	24	32	20	30	40	24	36	48

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[1] If D has latent roots + λ and − λ, and the corresponding invariant directions f_1 and f_2, then for D^2 all vectors $\alpha \underset{\sim}{f}_1 + \rho \underset{\sim}{f}_2$ will be latent vectors corresponding to the latent root λ^2. In an analogous fashion the invariant manifolds will be united in the case for which + λ and − λ are multiple latent roots. However, when the invariant manifolds of D^2 and the corresponding latent roots are known, the determination of them for D represents no particular difficulty. (See § 14, example 1).

§ 10 PROPERTIES OF THE INVARIANT MANIFOLDS OF CONJUGATE MATRICES

By using the methods indicated in the preceding paragraphs one can determine the dominant latent roots and the invariant manifolds constructed on axial and imaginary-axial manifolds corresponding to the latent roots found. At the same time, depending on the selection of the initial vector $\underset{\sim}{a}$, there may remain undiscovered individual latent roots with the same modulus, or the multiplicity of the latent roots found will not be fully determined. The latter will always occur if the elementary divisors of the matrix $D - \lambda E$ are not relatively prime.

Let the invariant manifold found have s dimensions; the aggregate of basic vectors for the simplest axial and imaginary-axial manifolds entering E_s forms a basis for E_s. We shall designate them by $\underset{\sim}{f}_1, \underset{\sim}{f}_2, \cdots, \underset{\sim}{f}_s$. We shall show how to pass from a matrix D to a new n - s-dimensional matrix D_*, defining a linear transformation in E_{n-s} and possessing the same latent roots as D, with the exception of those already found (taking into account the multiplicity).

To accomplish this we shall prove an auxiliary theorem concerning the matrix D_c, the conjugate of D [i.e., transpose - translator]. The elementary divisors of the matrix $D_c - \lambda E$ will obviously coincide with the elementary divisors of $D - \lambda E$, and accordingly the structure of the invariant manifolds of the matrices D and D_c will be identical. The connection between these manifolds is given by the following theorem.

Any axial invariant manifold of the matrix D_c corresponding to the latent root λ_i is orthogonal to all the invariant manifolds of the matrix D, with the exception of those which contain the axial manifolds corresponding to that particular latent root. Any m-dimensional axial invariant manifold of the matrix D_c is orthogonal to any k - m-dimensional invariant axial manifold of the matrix D that lies within whatsoever k-dimensional axial manifold corresponding to that particular latent root.

We shall first prove the first half of the theorem. For this it is sufficient to prove that the vectors $\underset{\sim}{g}_1, \underset{\sim}{g}_2, \cdots, \underset{\sim}{g}_m$ that form the fundamental basis of an arbitrary axial manifold of the matrix D_c corresponding to the latent root λ_i are orthogonal to the basic vectors $\underset{\sim}{h}_1, \underset{\sim}{h}_2, \cdots, \underset{\sim}{h}_k$ of an arbitrary axial manifold of the matrix D corresponding to a latent root $\lambda_j \neq \lambda_i$. The basis $\underset{\sim}{h}_1, \underset{\sim}{h}_2, \cdots, \underset{\sim}{h}_k$ we shall consider as being fundamental. From

$$\underset{\sim}{h}_1 \cdot D_c \underset{\sim}{g}_1 = \lambda_i \underset{\sim}{h}_1 \cdot \underset{\sim}{g}_1 = \underset{\sim}{g}_1 \cdot D \underset{\sim}{h}_1 = \lambda_j \underset{\sim}{g}_1 \cdot \underset{\sim}{h}_1$$

it follows that $\underset{\sim}{g}_1 \cdot \underset{\sim}{h}_1 = 0$. Furthermore, from

$$\underset{\sim}{g}_1 \cdot D \underset{\sim}{h}_\chi = (\lambda_j \underset{\sim}{h}_\chi + \underset{\sim}{h}_{\chi-1}) \cdot \underset{\sim}{g}_1 = \underset{\sim}{h}_\chi \cdot D_c \underset{\sim}{g}_1 = \lambda_i \underset{\sim}{h}_\chi \cdot \underset{\sim}{g}_1 \quad (\chi = 2, 3, \cdots, k)$$

there follow sequentially

$$\underset{\sim}{g}_1 \cdot \underset{\sim}{h}_2 = \underset{\sim}{g}_1 \cdot \underset{\sim}{h}_3 = \cdots = \underset{\sim}{g}_1 \cdot \underset{\sim}{h}_k = 0$$

In exactly the same way we find

$$\underset{\sim}{h}_1 \cdot \underset{\sim}{g}_2 = \underset{\sim}{h}_1 \cdot \underset{\sim}{g}_3 = \cdots = \underset{\sim}{h}_1 \cdot \underset{\sim}{g}_m = 0$$

Whence by the method of complete induction, on the basis of the equality

$$\mathbf{h}_\chi \cdot D_c\mathbf{g}_r - \mathbf{g}_r \cdot D\mathbf{h}_\chi = (\lambda_i - \lambda_j)\mathbf{g}_r \cdot \mathbf{h}_\chi + \mathbf{h}_\chi \cdot \mathbf{g}_{r-1} - \mathbf{h}_{\chi-1} \cdot \mathbf{g}_r = 0$$

it follows that

$$g_r \cdot h_\chi = 0 \qquad (r = 1,2,\cdots,m,\ \chi = 1,2,\cdots,k) \quad (10.1)$$

which is what was required to be proved.

The second part of the theorem is proved analogously. Let $\mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_m$ and $\mathbf{f}_1, \mathbf{f}_2, \cdots, \mathbf{f}_k$ now designate the vectors forming the fundamental basis of the axial invariant manifolds of the matrices D_c and D corresponding to the same latent root λ. From

$$\mathbf{g}_1 \cdot D\mathbf{f}_i = \mathbf{g}_1 \cdot (\lambda\mathbf{f}_i + \mathbf{f}_{i-1}) = \mathbf{f}_i \cdot D_c\mathbf{g}_1 = \lambda\mathbf{f}_i \cdot \mathbf{g}_1$$

it at once follows that

$$\mathbf{g}_1 \cdot \mathbf{f}_i = 0 \qquad (i = 1, 2, \cdots, k - 1)$$

Furthermore, on the ground that

$$\mathbf{g}_2 \cdot D\mathbf{f}_i = \mathbf{g}_2 \cdot (\lambda\mathbf{f}_i + \mathbf{f}_{i-1}) = \mathbf{f}_i \cdot D_c\mathbf{g}_2 = \mathbf{f}_i \cdot (\lambda\mathbf{g}_2 + \mathbf{g}_1)$$

we obtain

$$\mathbf{g}_2 \cdot \mathbf{f}_i = 0 \qquad (i = 1, 2, \cdots, k - 2)$$

Proceeding further, we will find that for any $\chi \leq m$

$$\underset{\sim}{g}_\chi \cdot \underset{\sim}{f}_i = 0 \qquad (i = 1, 2, \cdots, k - \chi) \quad (10.2)$$

which proves the second half of the theorem.

The proved theorem holds both for real and complex space, as is seen immediately from its proof.

Let now $\underset{\sim}{f}_1, \underset{\sim}{f}_2, \cdots, \underset{\sim}{f}_\chi$ and $\underset{\sim}{g}_1, \underset{\sim}{g}_2, \cdots, \underset{\sim}{g}_\chi$ be the fundamental system of generators for a certain maximum axial manifold E_χ of the matrix D and for the corresponding manifold E'_χ of the matrix D_c. By maximum we here understand such an axial invariant manifold as cannot be contained in an axial manifold of a greater number of dimensions.

The product $\underset{\sim}{g}_1 \cdot \underset{\sim}{f}_\chi$ will be different from zero. Let us assume at first that the elementary divisor $(\lambda - \lambda_i)^\chi$ to which the invariant manifolds correspond is relatively prime to all the rest of the elementary divisors. Then if it be assumed that $\underset{\sim}{g}_1 \cdot \underset{\sim}{f}_\chi = 0$, the vector $\underset{\sim}{g}_1$ would be orthogonal to n independent directions, which could not be, as $\underset{\sim}{g}_1 \neq 0$.

If now for the latent root λ_i we have r elementary divisors, given the vectors in hand $\underset{\sim}{f}_1, \underset{\sim}{f}_2, \cdots, \underset{\sim}{f}_\chi$, the vectors $\underset{\sim}{g}_1, \underset{\sim}{g}_2, \cdots, \underset{\sim}{g}_\chi$ will not be uniquely determined. In particular, for the vector $\underset{\sim}{g}_1$ there may be taken any vector of some r-dimensional manifold, and this selection made so that $\underset{\sim}{g}_1 \cdot \underset{\sim}{f}_\chi \neq 0$, which follows from considerations analogous to those presented above.

It is also easily shown that $\underset{\sim}{g}_{i+1} \cdot \underset{\sim}{f}_{\chi-i} \neq 0$ $(i = 1,2,\cdots,\chi-1)$. Assuming the contrary, in the simplest case we arrive immediately at the conclusion that $\underset{\sim}{g}_{i+1}$ may be linearly expressed in terms of $\underset{\sim}{g}_1, \cdots, \underset{\sim}{g}_i$, which is impossible on the ground of the linear independence of the vectors $\underset{\sim}{g}_1, \cdots, \underset{\sim}{g}_\chi$. In a case where the vector $\underset{\sim}{g}_{i+1}$ is not uniquely determined, it may always be chosen such that the demanded inequality is satisfied.

From the preceeding it follows that it is always possible to choose numbers c_{ik} such that the vectors

$$\bar{\underset{\sim}{g}}_i = \sum_{k=1}^{i} c_{ik}\underset{\sim}{g}_k \qquad (i = 1, 2, \cdots, \chi) \quad (10.3)$$

will form a system of vectors reciprocal to $\underset{\sim}{f}_\chi, \cdots, \underset{\sim}{f}_1$, that is, such that

$$\bar{\underset{\sim}{g}}_{i+1} \cdot \underset{\sim}{f}_{\chi-k} = \delta_{ik} \qquad (i,k = 0,1,\cdots,\chi-1) \quad (10.4)$$

Finding such a system of reciprocal vectors in each simplest invariant manifold of the matrix D_c, we shall obtain a complete system of vectors reciprocal to the system of n vectors composed from the fundamental basis of each simplest invariant manifold of the matrix D.

In the case of complex latent roots, the fulfilment of equation (10.4) for the complex vectors determining the corresponding invariant manifolds leads to the fulfilment of analogous equations for the real vectors forming the fundamental basis of the corresponding imaginary-axial manifolds.

§ 11. THE TRANSFORMATION OF THE GIVEN MATRIX D

Let us designate by $\underset{\sim}{g}_1{}', \cdots, \underset{\sim}{g}_s{}'$ the vectors corresponding to $\underset{\sim}{f}_1, \underset{\sim}{f}_2, \cdots, \underset{\sim}{f}_s$, selected from the system of reciprocal vectors constructed in the way indicated above.

Furthermore, we designate by α_i^j $(j = 1, 2, \cdots, n)$ the coordinates of the vector $\underset{\sim}{f}_i$, and by β_i^j $(j = 1, 2, \cdots, n)$ the coordinates of the vector $\underset{\sim}{g}_i{}'$.

We introduce into the space E_n new coordinate vectors, namely[1]

$$\begin{aligned} \underset{\sim}{e}_i{}' &= \underset{\sim}{e}_i - \beta_1^i \underset{\sim}{f}_1 - \beta_2^i \underset{\sim}{f}_2 - \cdots - \beta_s^i \underset{\sim}{f}_s \qquad (i = 1,2,\cdots,n-s) \\ \underset{\sim}{e}_{n-s+k}{}' &= \underset{\sim}{f}_k \qquad (k = 1,2,3,\cdots,s) \end{aligned} \tag{11.1}$$

This substitution of coordinates is equivalent to the transformation of the matrix D by means of a matrix T by the formula

$$D' = T^{-1} DT \tag{11.2}$$

Here T denotes a matrix whose elements t_{ik} are defined by the following equations:

$$\begin{aligned} t_{ik} &= \delta_{ik} - \sum_{\chi=1}^{s} \alpha_\chi^i \cdot \beta_\chi^k \qquad (i = 1,2, \cdots, n;\ k = 1,2, \cdots, n-s) \\ t_{i,n-s+k} &= \alpha_k^i \qquad (i = 1,2, \cdots, n;\ k = 1,2, \cdots, s) \end{aligned} \tag{11.3}$$

We designate the elements of the matrix T^{-1} by τ_{ik}.

---

[1] Here it is assumed, without detriment to the generality, that the determinant of the s-th order, $|\alpha_i^{n-s+k}|$, is different from zero.

On the basis of formulae (11.3) we readily find

$$\tau_{ik} = \delta_{ik} \qquad (i,k = 1,2, \cdots ,n-s)$$

$$\tau_{n-s+k,i} = \beta_k^{\ i} \qquad (i = 1,2, \cdots , n; k = 1, 2, \cdots , s)$$

$$\tau_{i,n-s+k} = - \frac{\begin{vmatrix} \alpha_1^{n-s+1} & \cdots & \alpha_1^{n-s+k-1} & \alpha_1^{\ i} & \alpha_1^{n-s+k+1} & \cdots & \alpha_1^{\ n} \\ \alpha_2^{n-s+1} & \cdots & \alpha_2^{n-s+k-1} & \alpha_2^{\ i} & \alpha_2^{n-s+k+1} & \cdots & \alpha_2^{\ n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_s^{n-s+1} & \cdots & \alpha_s^{n-s+k-1} & \alpha_s^{\ i} & \alpha_s^{n-s+k+1} & \cdots & \alpha_s^{\ n} \end{vmatrix}}{\begin{vmatrix} \alpha_1^{n-s+1} & \alpha_1^{n-s+2} & \cdots & \alpha_1^{\ n} \\ \alpha_2^{n-s+1} & \alpha_2^{n-s+2} & \cdots & \alpha_2^{\ n} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_s^{n-s+1} & \alpha_s^{n-s+2} & \cdots & \alpha_s^{\ n} \end{vmatrix}} \qquad (11.4)$$

The elements of the matrix D', transformed on the basis of (11.2), may be computed by the formula

$$d_{ik}' = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \tau_{i\alpha} d_{\alpha\beta} t_{\beta k} \qquad (11.5)$$

Substituting in the expressions exhibited above for $\tau_{i\alpha}$ and $\tau_{\beta k}$, we arrive at the following results:

1) $d_{ij}' = d_{ij} + \sum\limits_{\alpha=n-s+1}^{n} \tau_{i\alpha} d_{\alpha j} \quad (i,j = 1,2,\cdots,n-s)$ (11.6)

2) $d_{ij}' = 0 \quad (i = 1,2,\cdots,n-s;\ j = n-s+1,\cdots,n)$ (11.7)

3) The s-dimensional matrix with elements $d_{ij}'(i,j > n-s)$ will be the matrix, in Jordan canonical form, of the transformation induced in the invariant manifold E_s.

4) If some axial manifold E_r present in E_s and corresponding to the latent root λ, with basis $\underset{\sim}{f}_{m+1}, \cdots, \underset{\sim}{f}_{m+r}$ is not maximum — that is, there exists outside of E_s a vector $\underset{\sim}{f}$ such that $D\underset{\sim}{f} = \lambda \underset{\sim}{f} + \underset{\sim}{f}_{m+r}$, — then the element d_{ij}' for $i = n-s+m+r$ and $j \leq n-s$ will coincide with the j-th contravariant coordinate (with respect to the new coordinate vectors) of the vector $\underset{\sim}{g}$ corresponding to $\underset{\sim}{f}$ in the reciprocal system of vectors.

For other values $i > n-s$ and $j \leq n-s$, $d_{ij}' = 0$.

Points 2, 3, and 4 may even be demonstrated without computation by formula (11.5), and directly on the strength of the properties of the fundamental bases of the simplest invariant manifolds and of formula (11.1) with the coordinate vectors replaced.

§12. THE INVARIANT MANIFOLDS OF THE TRANSFORMED MATRIX.

Let us consider the $n-s$-dimensional manifold E_{n-s} constructed on the vectors $\chi_1', \cdots, \chi_{n-s}'$. This manifold does not have common directions with E_s; accordingly together with E_s it completely defines the whole space. The matrix D_* with elements d_{ij}' $(i,j \leq n-s)$ defines a transformation induced by matrix D in E_{n-s}. Any vector belonging to E_{n-s} has its last s coordinates (in E_n) equal to zero. Its first $n-s$ coordinates are transformed by the matrix D_* just as they are by matrix D'. Consequently any axial manifold E_χ of the matrix D not contained in E_s will lie in E_{n-s} and will there be an invariant axial manifold of the matrix D_*.

If a certain axial manifold E_r belonging to E_s is part of an axial manifold of a greater number of dimensions, E_χ, the matrix D_* will have the invariant axial manifold $E_{\chi-r}$ corresponding to the same latent root.

Indeed, let $\underset{\sim}{h}_1, \cdots, \underset{\sim}{h}_\chi$ be a fundamental basis in E_χ. Then $\underset{\sim}{h}_1, \cdots, \underset{\sim}{h}_r$, under the condition, will coincide with certain of the vectors $\underset{\sim}{f}_i$. Let $\underset{\sim}{h}_i = \underset{\sim}{f}_{m+i}$ $(i = 1, 2, \cdots, r)$. Consider the vector $\underset{\sim}{a} = c_1\underset{\sim}{h}_{r+1} + c_2\underset{\sim}{h}_{r+2} + \cdots + c_{\chi-r}\underset{\sim}{h}_\chi$, where $c_1, \cdots, c_{\chi-r}$ are arbitrary numbers. Then

$$D\underset{\sim}{a} = \lambda\underset{\sim}{a} + c_2\underset{\sim}{h}_{r+1} + c_3\underset{\sim}{h}_{r+2} + \cdots + c_{\chi-r}\underset{\sim}{h}_{\chi-1} + c_1\underset{\sim}{h}_r \tag{12.1}$$

In the new system of coordinates the last s coordinates of the vector $\underset{\sim}{a}$ are equal to zero, since $\underset{\sim}{h}_{r+1}, \cdots, \underset{\sim}{h}_\chi$, according to the condition, lie in E_{n-s}. The vector $\underset{\sim}{h}_r$ has all coordinates equal to zero except the $(n-s+m+r)$-th. Since the matrix D_* transforms the first $n-s$ coordinates just as does D', it thence follows that

$$D_*\underset{\sim}{a}_* = \lambda\underset{\sim}{a}_* + c_2\underset{\sim}{h}_{*r+1} + \cdots + c_{\chi-r}\underset{\sim}{h}_{*\chi-1} \tag{12.2}$$

where $\underset{\sim}{a}_*, \underset{\sim}{h}_{*r+1}, \cdots, \underset{\sim}{h}_{*\chi-1}$ signify the $n-s$ -dimensional vectors the coordinates of which coincide with the first $n-s$ coordinates of the vectors $\underset{\sim}{a}, \underset{\sim}{h}_{r+1}, \cdots, \underset{\sim}{h}_{\chi-1}$. It follows from equation (12.2) that the vectors $\underset{\sim}{h}_{*r+1}, \cdots, \underset{\sim}{h}_{*\chi}$ are a fundamental basis of the axial manifold $E_{\chi-r}$ of the matrix D_*. The vector $\underset{\sim}{h}_{*r+1}$ is the axis of this manifold.

§13. FURTHER DETERMINATION OF THE LATENT ROOTS AND THE INVARIANT MANIFOLDS OF THE MATRIX D

If a certain invariant manifold E_s and the corresponding latent roots have been determined, one can go over to a matrix of $n - s$ dimensions to reveal the remaining latent roots and invariant manifold $E_{s'}$ of the matrix D_c. This may be done using the very method

by which E_s was found. We only remark that to make possible the future determination of the vectors $\bar{\underset{\sim}{g}}_i$ it is necessary that the axial invariant manifolds entering into $E_{s'}$ be certainly maximum.

It is always possible to attain this by taking as the initial vector during the determination of $E_{s'}$ a linear combination of vectors directed along the invariant directions already found, lying in E_s; (given the presence in E_s of imaginary-axial manifolds, the first two generating vectors of the fundamental system are to be taken from them). Then on the strength of the results obtained in §10, this initial vector will have components different from zero, lying in the maximum axial manifolds of the matrix D_c, which manifolds correspond to those axial manifolds of the matrix D from which we have taken the axial directions. Therefore, generally speaking, $s' \geq s$.

Further transformations are to be conducted as indicated in §10, 11.

During this, since for D_c the dominant latent roots are known, as soon as the vectors $D_c^m a$, $D_c^{m+1} a$, ... , are computed, the vectors forming the fundamental basis for $E_{s'}$ may be found at once by the corresponding formulae.

We now remark that in formula (11.6), by which the elements of the matrix D_* are computed, the coordinates of the vectors $\underset{\sim}{g}_i$ do not appear. Its [D_*] latent roots and invariant manifolds may accordingly be found even without the computation of the vectors $\underset{\sim}{g}_i$.

However to obtain the invariant manifolds of the matrix Γ one must return to the original coordinate system, which cannot be done unless one has the coordinates of the vectors $\underset{\sim}{g}_i$.

In this manner, after the determination, by the method indicated in the first half of the present work, of the invariant manifold E_s of the matrix D, which manifold is composed from the axial or imaginary-axial simplest manifolds corresponding to the dominant latent roots, we construct the matrix D_* with elements d_{*ik} which are defined by formula (11.6). The latent roots of this $n-s$-dimensional matrix coincide with $n-s$ (with due regard for multiplicity) as yet unfound latent roots of the matrix D. By the same method the dominant latent roots of the matrix D_* and the corresponding invariant manifolds can be determined.

If we want to obtain these manifolds in the original system of coordinates, we must carry through the process indicated in §12 for determining the vectors $\underset{\sim}{g}_i'$. To obtain the coordinates of any vector in the original system of coordinates the formula

$$a_k = \sum_{i=1}^{n-s} t_{ik}\, a_{*i} \qquad (k = 1, 2, \cdots, n) \qquad (13.1)$$

will obviously serve. Here a_{*i} signifies the coordinates of the corresponding vector in E_{n-s}, and t_{ik} are determined by formula (11.3).

Continuing in this manner, we shall pass to matrices of ever lower order, gradually determining all the latent roots of the matrix D.

14. NUMERICAL EXAMPLES

Example 1. Let it be required to determine the latent roots and invariant directions of the matrix

$$D = \begin{Bmatrix} 4 & -6 & 0 & -4 \\ -6 & 4 & 4 & 0 \\ 0 & 4 & 4 & 6 \\ -4 & 0 & 6 & 4 \end{Bmatrix}$$

For expediting the convergence we shall carry out the iterations not with D but with D^2:

$$D^2 = \begin{Bmatrix} 68 & -48 & -48 & -32 \\ -48 & 68 & 32 & 48 \\ -48 & 32 & 68 & 48 \\ -32 & 48 & 48 & 68 \end{Bmatrix}$$

We shall take for the initial vector $\underset{\sim}{a} = \{1, 0, 0, 0\}$.

In Table 1 are given the vectors $\underset{\sim}{a}_k = \mu_k D^{2k} \underset{\sim}{a}$; the computations terminate at $k = 8$, since the ratio of the coordinates of the vectors $\underset{\sim}{a}_8$ and $\underset{\sim}{a}_7$ coincide. The magnitude of this ratio gives $\lambda_1{}^2 = 196.000$.

Table 1

k	μ_k	$\underset{\sim}{a}_k$			
1	1	68	-48	-48	-32
2	0.1	978	-960	-960	-896
3	0.01	1873	-1860	-1860	-1844
4	0.1	36493	-36442	-36442	-36389
5	0.01	71444	-71425	-71425	-71407
6	0.01	140000	-139993	-139993	-139987
7	0.01	274389	-274387	-274387	-274384
8	0.01	537799	-537798	-537798	-537797

A direct check immediately convinces that $\underset{\sim}{f}_1$ (1, -1, -1, -1) defines the invariant direction of D corresponding to the latent root $\lambda_1 = 14$.

For the determination of the remaining latent roots we pass to three-dimensional space; in this case formulae (11.6) will yield

$$d_{*ik} = d_{ik} - \frac{\alpha_i}{\alpha_4} d_{4k} \qquad (i,k = 1, 2, 3)$$

where α_1, α_2, α_3, α_4 are the coordinates of $\underset{\sim}{f}$. Performing the computation, we obtain

$$D_* = \left\{ \begin{matrix} 36 & 0 & 0 \\ -16 & 20 & -16 \\ -16 & -16 & 20 \end{matrix} \right\}$$

Taking for the initial vector $\underset{\sim}{b} = \left\{1, 0, 0\right\}$, we compute $\underset{\sim}{b}_k = \mu_k D_*^k \underset{\sim}{b}$ (Table 2) up to $k = 6$, when the ratio of coordinates gives $\lambda_2^2 = 36.000$.

Table 2

k	μ_k	$\underset{\sim}{b}_k$		
1	1	36	-16	-16
2	1	1296	-640	-640
3	0.1	4666	-2330	-2330
4	0.1	16798	-8398	-8398
5	0.1	60473	-30236	-30236
6	0.1	217703	-108851	-108851

For the coordinates of the latent vector $\underset{\sim}{f}_{*2}$ there can be adopted

$$\alpha_{*2}^1 = 1, \quad \alpha_{*2}^2 = -\tfrac{1}{2}, \quad \alpha_{*2}^3 = -\tfrac{1}{2}$$

The presence of two zeros in the first row of D_* at once indicates that for D_{*c} the latent vector corresponding to $\lambda_2^2 = 36$

will be $\underset{\sim}{g}_{*c}$ (1, 0, 0). Passing on to a matrix of the second order by formulae (11.6), we obtain

$$D_{**} = \begin{Bmatrix} 4 & -32 \\ 0 & 36 \end{Bmatrix}$$

Direct computations will give the latent roots $\lambda_3^{\,2} = 36$ and $\lambda_4^{\,2} = 4$, and the corresponding invariant vectors $\underset{\sim}{f}_{**3}(1, -1)$ and $\underset{\sim}{f}_{**4}(1, 0)$.

In order to obtain the invariant directions of the given matrix one must perform a reverse transition to the space of the greatest number of dimensions. First we find the invariant directions D_* using formula (13.1):

$$\alpha^{\gamma}_{*k} = \sum_{i=1,2} t_{*\gamma i}\,\alpha^{i}_{**k} \qquad (k = 3,4;\ \gamma = 1,2,3)$$

here $t_{*\gamma i} = \delta_{\gamma i} - \alpha^{\gamma}_{*2}\,\beta^{i}_{*2}$, and β^{i}_{*2} are the coordinates of the vector $\underset{\sim}{g}_{*2}$, normalized so that $\underset{\sim}{f}_{*2}\cdot\underset{\sim}{g}_{*2} = 1$.

The computations give

$$t_{11} = t_{12} = t_{32} = 0\,, \quad t_{21} = t_{31} = 0.5\,, \quad t_{22} = 1$$

whence we find

$$\underset{\sim}{f}_3 = \{0, -0.5, 0.5\}\,; \quad \underset{\sim}{f}_4 = \{0, 0.5, 0.5\}$$

For the further transition to four-dimensional space we are served by the formulae:

$$\alpha_k^{\gamma} = \sum_{i=1,2,3} t_{\gamma i}\,\alpha^{i}_{*k} \qquad (k = 2,3,4;\ \gamma = 1,2,3,4)$$

where $t_{\chi i} = \delta_{\chi i} - \alpha_1^{\chi} \alpha_1^{i}$, and α_1^{i} is the coordinate of $\underset{\sim}{f}_1$, which must be normalized so that $\underset{\sim}{f}_1 \cdot \underset{\sim}{f}_1 = 1$. Performing the computations, for convenience replacing $\underset{\sim}{f}_{*3}$ by $2\underset{\sim}{f}_{*3}$ and $\underset{\sim}{f}_{*4}$ by $4\underset{\sim}{f}_{*4}$, we obtain:

$$\begin{array}{lll} t_{11} = 0.75 & t_{12} = 0.25 & t_{13} = 0.25 \\ t_{21} = 0.25 & t_{22} = 0.75 & t_{23} = -0.25 \\ t_{31} = 0.25 & t_{32} = -0.25 & t_{33} = 0.75 \\ t_{41} = 0.25 & t_{42} = -0.25 & t_{43} = -0.25 \end{array}$$

$$\underset{\sim}{f}_2 = \{-1, 0, 0, -1\}, \quad \underset{\sim}{f}_3 = \{0, -1, 1, 0\}, \quad \underset{\sim}{f}_4 = \{1, 1, 1, -1\}$$

For control and the determination of the signs of λ_i, we compute $D\underset{\sim}{f}_i$:

$$D\underset{\sim}{f}_2 = \{0, 6, -6, 0\}, \quad D\underset{\sim}{f}_3 = \{6, 0, 0, 6\}, \quad D\underset{\sim}{f}_4 = \{2, 2, 2, -2\}.$$

From the results it is obvious that $\lambda_4 = 2$ and that the invariant plane for D^2, each direction of which plane is invariant, contains only two invariant directions corresponding to the latent roots $\lambda_2 = 6$ and $\lambda_3 = -6$; (see the last footnote to §9). As the vectors defining these invariant directions we may take

$$D\underset{\sim}{f}_2 + 6\underset{\sim}{f}_2 = \{-6, 6, -6, -6\}, \quad D\underset{\sim}{f}_2 - 6\underset{\sim}{f}_2 = \{6, 6, -6, 6\}.$$

Example 2. Let it be required to determine the latent roots and invariant manifolds of the matrix

$$D = \begin{Bmatrix} 6 & -3 & 4 & 1 \\ 4 & 2 & 4 & 0 \\ 4 & -2 & 3 & 1 \\ 4 & 2 & 3 & 1 \end{Bmatrix}$$

We shall iterate the matrix

$$D^2 = \left\{ \begin{matrix} 44 & -30 & 27 & 11 \\ 48 & -16 & 36 & 8 \\ 32 & -20 & 20 & 8 \\ 48 & -12 & 36 & 8 \end{matrix} \right\}$$

having adopted for the initial vector $\underset{\sim}{a} \{1, 0, 0, 0\}$. The vectors $\underset{\sim}{a}_k = \mu_k D^{2k} \underset{\sim}{a}$ are set forth in Table 3.

Table 3

k	μ_k	$\underset{\sim}{a}_k$			
1	1	44	48	32	48
2	1	1888	2880	1472	3072
3	1	70208	122112	56832	133632
4	0.01	24302	45312	120101	50196
5	1	804811	1566708	675012	1747956
6	1	25863284	51847680	21903680	58114512
7	0.1	81321309	166532333	69366118	187271405

The vectors $\underset{\sim}{a}_k$ turn out to be nearly of the same direction, but they do not tend so rapidly to the limiting invariant direction as in the preceeding example, the which compels us to surmise that multiple roots are present (case 1 of those considered in § 7). Actually the coordinates of the polyvectors $[\underset{\sim}{a}_5 \underset{\sim}{a}_6]$, $[\underset{\sim}{a}_5 \underset{\sim}{a}_7]$, $[\underset{\sim}{a}_5 \underset{\sim}{a}_7]$ prove to be approximately proportional and their ratio gives, by formulae (4.8) and (4.9) for $\chi = 2$

$$\frac{\lambda_1^2}{2} = \frac{[\underset{\sim}{a}_6 \underset{\sim}{a}_7]}{[\underset{\sim}{a}_5 \underset{\sim}{a}_7]} = 13.7081, \quad 2\lambda_1^2 = \frac{[\underset{\sim}{a}_5 \underset{\sim}{a}_7]}{[\underset{\sim}{a}_5 \underset{\sim}{a}_6]} = 54.8323$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

whence $\lambda_1^2 = 27.4162$, and the values obtained from both formulae coincide sufficiently well to indicate the correctness of the surmise made about the multiplicity of the root. [See Note 3 at the end of the translation.]

The values of λ_1^2 obtained may be easily verified and made more precise. Actually the vector $\underset{\sim}{q} = (D^2 - \lambda_1^2)\underset{\sim}{a}_6$ ought to give the invariant direction. The computation shows that the ratio of the coordinates of $D^2\underset{\sim}{q}$ and $\underset{\sim}{q}$ is in fact constant, but equal to 27.4166. If the arithmetic mean of both values obtained — 27.4164 — be taken for λ_1^2, and $\underset{\sim}{q}$ and $D^2\underset{\sim}{q}$ again be found, we will obtain for the ratio of coordinates 27.41642. Again taking the arithmetic mean — 27.41641 — for λ_1^2, we shall obtain for the ratio the quantity 27.416406. Limiting ourselves to five figures of decimals, $\lambda_1^2 = 27.41641$ may be adopted. For the vector $\underset{\sim}{f}_1(\alpha_1^{\,1}, \alpha_1^{\,2}, \alpha_1^{\,3}, \alpha_1^{\,4})$ defining the invariant direction, there may be taken the vector $\underset{\sim}{q}$, the first coordinate of which has been reduced to unity; and for the vector $\underset{\sim}{f}_2(\alpha_2^{\,1}, \alpha_2^{\,2}, \alpha_2^{\,3}, \alpha_2^{\,4})$, defining together with $\underset{\sim}{f}_1$ the axial invariant manifold E_2, the vector $\underset{\sim}{a}_7$, normalized likewise:

$$\underset{\sim}{f}_1 = (1.0000000,\ \ 2.3416411,\ \ 0.8944272,\ \ 2.6832819)$$

$$\underset{\sim}{f}_2 = (1.0000000,\ \ 2.0478314,\ \ 0.8529882,\ \ 2.3028576)$$

Formulae (11.6) for the transition to two-dimensional space reduce, in the case in hand, to the form

$$d_{*ij} = d_{ij} + \sum_{\alpha=3,4} \tau_{i\alpha} d_{\alpha j} \qquad (i,j = 1,2)$$

Here

$$\tau_{ik} = \frac{(-1)^{k-1}\begin{vmatrix}\alpha_1^{\,i} & \alpha_1^{\,k}\\ \alpha_2^{\,i} & \alpha_2^{\,k}\end{vmatrix}}{\begin{vmatrix}\alpha_1^{\,3} & \alpha_1^{\,4}\\ \alpha_2^{\,3} & \alpha_2^{\,4}\end{vmatrix}} \qquad (k = 3,4;\ i = 1,2)$$

The computations give

$$D_* = \left\{\begin{matrix} -0.460365 & 1.043957 \\ -1.043965 & 1.627556 \end{matrix}\right\}$$

The characteristic equation for D_* will be

$$\lambda^2 - 1.167191\,\lambda + 0.340585 = 0\ ,$$

both roots of which are equal to 0.583596. The invariant direction of the matrix D_* for this latent root is defined by the vector (1.043961, 1.043957), which can with sufficient accuracy be replaced by the vector $\underset{\sim}{f}_{*3}(1,\ 1)$. Any other vector linearly independent of $\underset{\sim}{f}_{*3}$, even $\underset{\sim}{f}_{*4}(0,\ 1)$, defines together with $\underset{\sim}{f}_{*3}$ the corresponding axial manifold.

To obtain the vectors $\underset{\sim}{f}_3$ and $\underset{\sim}{f}_4$ in four-dimensional space, the invariant manifolds of the matrix $D_c^{\,2}$ must be found. The computations $\underset{\sim}{b}_k = \mu_k D_c^{\,2k} \underset{\sim}{b}$ are given in Table 4. For the initial vector $\underset{\sim}{b}$ the vector (1, 2, 1, 2), approximately coincident in direction with $\underset{\sim}{f}_1$ has been taken. (See above.)

Table 4

k	μ_k	$\underset{\sim}{b}_k$			
1	1	268	-106	191	51
2	1	15264	-10776	9076	4036
3	1	638528	-515456	351008	186592
4	1	23542016	-20167808	12421312	7200960
5	0.01	8109220	- 7184133	4172541	2545979
6	0.1	26769560	-24233304	13542622	8547652
7	0.1	85831325	-78877820	42894709	27732092
8	0.1	269421404	-250357391	133409375	87813542

The invariant direction of $D_c^{\,2}$ is defined by the vector $\underset{\sim}{h}_1 = (D^2 - \lambda_1^{\,2})\underset{\sim}{b}_7$, and for the vector defining the corresponding axial invariant manifold, $\underset{\sim}{h}_2 = \underset{\sim}{b}_7$ may be taken. From these vectors we construct the vectors

$$\underset{\sim}{g}_1(\rho_1^{\,1}, \rho_1^{\,2}, \rho_1^{\,3}, \rho_1^{\,4}) \quad \text{and} \quad \underset{\sim}{g}_2(\rho_2^{\,1}, \rho_2^{\,2}, \rho_2^{\,3}, \rho_2^{\,4}) \quad ,$$

which are reciprocal with $\underset{\sim}{f}_1$ and $\underset{\sim}{f}_2$, by the formulae

$$\underset{\sim}{g}_2 = \frac{\underset{\sim}{h}_1}{\underset{\sim}{h}_1 \cdot \underset{\sim}{f}_2} , \quad \underset{\sim}{g}_1 = - \frac{\underset{\sim}{h}_1(\underset{\sim}{h}_2 \cdot \underset{\sim}{f}_2)}{(\underset{\sim}{h}_1 \cdot \underset{\sim}{f}_2)(\underset{\sim}{h}_2 \cdot \underset{\sim}{f}_1)} + \frac{\underset{\sim}{h}_2}{(\underset{\sim}{h}_2 \cdot \underset{\sim}{f}_1)}$$

The computations give

$$\underset{\sim}{g}_1 = (-6.2611964, \quad 6.7611970, \quad -2.6786166, \quad -2.3013866) \ ,$$
$$\underset{\sim}{g}_2 = (\ 6.9847964, \quad -6.987964, \quad 3.2376309, \quad 2.4131876) \ .$$

Further, we find t_{1k} by formulae (11.3) and the coordinates of $\underset{\sim}{f}_3$ and $\underset{\sim}{f}_4$ by formulae (13.1):

$$t_{11} = 0.276400,\ t_{21} = 0.357789,\ t_{31} = -0.357764,\ t_{41} = 0.715564$$

$$t_{12} = 0.223600,\ t_{22} = -0.528611,\ t_{32} = -0.08945,\ t_{42} = -2.057206$$

$$\underset{\sim}{f}_3 = (0.500000, \quad -0.170822, \quad -0.447214, \quad -1.341642)$$

$$\underset{\sim}{f}_4 = (0.223600, \quad -0.528611, \quad -0.089450, \quad -2.057206)$$

A control computation $D\underset{\sim}{f}_3$ gives for the ratio of the coordinates of $D\underset{\sim}{f}_3$ and $\underset{\sim}{f}$ the numbers: 0.76393, 0.76396, 0.76393, 0.76394, which are in full agreement with the value found earlier for $\lambda_3^{\,2} = 0.583596$.

Example 3. To be determined, the invariant directions and latent roots of the matrix

$$D = \begin{pmatrix} 15 & 11 & 6 & -9 & -15 \\ 1 & 3 & 9 & -3 & -8 \\ 7 & 6 & 6 & -3 & -11 \\ 7 & 7 & 5 & -3 & -11 \\ 17 & 12 & 5 & -10 & -16 \end{pmatrix}$$

The computations $\underset{\sim}{a}_k = \mu_k D^k \underset{\sim}{a}$ for $\underset{\sim}{a} = (1, 1, 1, 1, 1)$ are given in Table 5. The multiplier is $\mu_k = 0.1$ for $k = 14-16$, and $\mu_k = 1$ for all the other values of k.

Table 5

k	$\underset{\sim}{a}_k$				
0	1	1	1	1	1
1	8	2	5	5	8
2	7	-20	-5	-8	7
3	-178	-130	-154	-169	-178
4	-833	-23	-485	-461	-818
5	761	2660	1502	1964	836
6	19471	9679	15211	16369	19352
7	52214	-18508	23669	18137	49904
8	-190157	-243932	-206891	-249068	-195947
9	-1596136	-469192	-1133416	-1170457	-1568476
10	-1842395	2854723	- 23993	- 640231	-1631510
11	22332641	17637224	20113532	-22992248	22517846
12	104982349	7146589	66159973	63693665	101661229
13	- 47763868	-282478954	-134608699	-193622083	- 62157448
14	-195641824	-102855319	-157227482	-172014507	-194403670
15	-545215148	152017762	-275505683	-221133520	-515708456
16	156676258	222035520	177876001	220628346	164069771

The course of the computations points to the presence of complex roots, and the lack of proportionality of the coordinates of the bivectors $[\underset{\sim}{a}_k \underset{\sim}{a}_{k+1}]$ points to an equality of the moduli of more than two latent roots. For determining them we shall employ the formulae of § 8. Having taken $\underset{\sim}{q}_o = \underset{\sim}{a}_{12}$, we compute one after another the coordinates of the polyvectors $[\underset{\sim}{q}_o \underset{\sim}{q}_1]$, $[\underset{\sim}{q}_1 \underset{\sim}{q}_2]$, $\cdots$; $[\underset{\sim}{q}_o \underset{\sim}{q}_1 \underset{\sim}{q}_2]$, $[\underset{\sim}{q}_1 \underset{\sim}{q}_2 \underset{\sim}{q}_3]$, $\cdots$.

Then as the ratios of the coordinates of the bivectors coincide, within the limits of the first two or three significant figures, the trivectors turn out to be non-proportional. Finally, two coordinates of four different vectors are computed; these are determinants of the

fourth order, composed of the first, second, third, fourth, and of the second, third, fourth and fifth coordinates of the corresponding vectors. The results of the computations are brought together in Table 6.

Table 6

Polyvector	First Coordinate	Second Coordinate
$[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_3]$	1.888421×10^{22}	-2.360593×10^{22}
$[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_4]$	1.133201	-1.416486
$[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_3\underset{\sim}{q}_4]$	7.365237	-9.206640
$[\underset{\sim}{q}_0\underset{\sim}{q}_2\underset{\sim}{q}_3\underset{\sim}{q}_4]$	1.699807	-2.124734
$[\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_3\underset{\sim}{q}_4]$	4.248808	-5.311186

The ratios of the coordinates coincide sufficiently well; (it must be borne in mind that a loss of accuracy occurs during the computation of the determinants, owing to the subtraction of numbers little different).

On the basis of formulae (8.4) we find (the means of the two values are everywhere adopted):

$$\rho^4=\frac{[\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_3\underset{\sim}{q}_4]}{[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_3]}=224.994;\quad \rho^2=\frac{[\underset{\sim}{q}_0\underset{\sim}{q}_2\underset{\sim}{q}_3\underset{\sim}{q}_4]}{[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_4]}=15.000$$

$$2\rho(\cos\phi_1+\cos\phi_2)=\frac{[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_4]}{[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_3]}=6.00068;\quad 2\rho^2(1+\cos\phi_1\cos\phi_2)=\frac{[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_3\underset{\sim}{q}_4]}{[\underset{\sim}{q}_0\underset{\sim}{q}_1\underset{\sim}{q}_2\underset{\sim}{q}_3]}=39.0018$$

whence

$$\rho=3.87297,\ \cos\phi_1=\cos\phi_2=0.38734,\ \lambda_{1,2,3,4}=1.50016\pm i3.57064$$

The vectors forming the fundamental basis of the corresponding imaginary-axial manifold may readily be found, as was shown at the end of § 9.

The fifth latent root may at once be obtained from the trace (spur) of the matrix:

$$s = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 15 + 3 + 6 - 6 - 16 = 5; \ \lambda_5 = -1.00032$$

The corresponding invariant direction will be defined by the vector[1]

$$\underset{\sim}{f}_5 = (D^2 - 2\rho \cos \phi D + \rho^2)^2 \underset{\sim}{a} = \{13.30, 22.05, 19.09, 16.09, 28.12\}$$

By a direct check it is readily confirmed that the vector $\{13, 22, 19, 16, 28\}$ exactly defines the invariant direction corresponding to the latent root $\lambda_5 = -1.00000$.

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[1] In the work of Aitken [4], whence the matrix for the present example has been taken, the vector $\underset{\sim}{f}_5$ is given incorrectly, without any indication whatsoever of the manner in which it was obtained. There are also given the exact value $\rho = \sqrt{15}$ and $\cos \phi = \sqrt{0.15}$.

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Notes

by editor of translation

1. For a definition and discussion of polyvectors (or multivectors) in the restricted sense used here, see Élie Cartan, La Géométrie des espaces de Riemann, Mém. des sci. math., no. 9, Paris, 1925, page 4. We may summarize the definition briefly as follows:

Given the k column vectors $\underset{\sim}{q}_1, \underset{\sim}{q}_2, \cdots, \underset{\sim}{q}_k$ in n-dimensional Euclidean space, form the matrix Q of n rows and k columns, whose i-th column consists of the n components of $\underset{\sim}{q}_i$ $(i = 1, 2, \cdots, k)$. The polyvector $[\underset{\sim}{q}_1, \underset{\sim}{q}_2, \cdots, \underset{\sim}{q}_k]$ is a $\binom{n}{k}$-dimensional array whose components are the k-dimensional minor determinants of Q. The components of a multivector are not necessarily independent functions of the vectors $\underset{\sim}{q}_1, \cdots, \underset{\sim}{q}_k$.

One may easily prove the following properties of polyvectors, which are used in proving (4.9):

i) If any $\underset{\sim}{q}_i = 0$, $[\underset{\sim}{q}_1, \cdots, \underset{\sim}{q}_k] = 0$.

ii) If any two $\underset{\sim}{q}$'s are proportional, $[\underset{\sim}{q}_1, \cdots, \underset{\sim}{q}_k] = 0$.

iii) $[\underset{\sim}{q}_1, \cdots, \underset{\sim}{q}_k]$ is a linear function of each single one of its arguments $\underset{\sim}{q}_1, \cdots, \underset{\sim}{q}_k$.

When k is 2 or 3, the author uses the terms bivector and trivector, respectively.

2. Probably the author meant to say that the ratio will grow when $|\lambda_i| > |\lambda_1|$, except when $\underset{\sim}{a}_i$ coincides with the invariant direction.

3. Here and later the author divides one polyvector by another, according to the following rule: Given two proportional vectors $\underset{\sim}{a}, \underset{\sim}{b}$, with $\underset{\sim}{b} \neq 0$ and $\underset{\sim}{a} = \alpha \underset{\sim}{b}$ (α = scalar), the quotient $\underset{\sim}{a}/\underset{\sim}{b}$ is defined to be α.

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